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Mir Publishers Moscow

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ОСНОВЫ РЕЛЯТИВИСТСКОЙ ТЕОРИИ ГРАВИТАЦИИ

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TO THE READER

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Preface

In this book we give a detailed exposition of the relativistic theory of gravitation, or RTG, developed in Logunov, 1986, Logunov and Mestvirishvili, 1984, 1985a, 1985b, 1986b, Vlasov and Logunov, 1984, and Vlasov, Logunov, and Mestvirishvili, 1984. In these works RTG has been built unambiguously, using as a basis the relativity principle, the gauge principle, and the geometrization principle. The gravitational field is constructed as a physical field in the spirit of Faraday and Maxwell, and this field has energy, momentum, and spins 2 and 0. RTG revives the concept of a classical gravitational field that no choice of reference frame can destroy since it is a material substratum. The gauge principle is formulated on the basis of the local infinite dimensional noncommutative group of supercoordinate transformations.

The system of RTG equations (8.28) and (8.29), is general-covariant and depends explicitly on the metric tensor of the Minkowski space-time. This unambiguously separates the forces of inertia from the gravitational field. All field variables in the RTG equations are functions of space-time coordinates of the Minkowski space-time.

The theory considered here rigorously obeys the laws of conservation of energy-momentum and angular momentum for matter and gravitational field taken together. It also describes the entire body of gravitational experiments. We show that Einstein's formula for gravitational waves, (15.56), follows directly from the theory. In analyzing the evolution of the universe, RTG concludes that the universe is infinite and "flat" and predicts a large "latent" mass in it. This "latent" mass exceeds the observable mass of the universe by a factor of 40.

RTG predicts that gravitational collapse, which for the comoving observer occurs over a finite proper time interval, does not lead to the infinite contraction of matter; rather, it terminates when the object achieves a certain finite density, and the physical processes associated with the object slow down indefinitely as the object approaches the radius equal to GM. This means that in RTG there can be no static or nonstatic spherically symmetric objects with a radius equal to or less than GM. From the standpoint of an external observer the luminosity of a collapsing object decreases exponentially (it blackens), but nothing unusual happens to it. Hence, according to this approach, there can be no objects in nature in which the gravitational contraction of matter results in infinite density (black holes). However, objects can exist that have an extremely large mass and an inner structure.

We also show that in general relativity, GR, there are no fundamental laws of conservation of energy-momentum and angular momentum of matter and gravitational field taken together, with the result that the inertial mass defined in GR is not equal to the active gravitational mass. We have established that GR gives no definite predictions concerning gravitational effects. Finally, in GR the gravitational field is not a physical field possessing an energy-momentum density. Consequently, Einstein's formula (15.56) for gravitational waves does not follow from GR.

Our colleagues A. A. Vlasov, Yu. M. Loskutov, and Yu. V. Chugreev contributed considerably to the development of some aspects of RTG incorporated in this book, and Chapter 20 was written with the aid of Yu. V. Chugreev. To all of them our sincere thanks.

A. A. Logunov
M. A. Mestvirishvili

Introduction

Before presenting the basics of the relativistic theory of gravitation (RTG) proposed here, we will touch briefly on some important aspects of the general relativity (GR) theory.

In creating his general relativity theory, Einstein proceeded from the principle of equivalence of the forces of inertia and gravity. The equivalence principle was formulated in the following manner (Einstein, 1925):

> ...for any infinitely small world-region the coordinates can always be chosen in such a way that the gravitational field in that world-region vanishes.

In formulating the equivalence principle Einstein already departed from the idea of a gravitational field as being a Faraday-Maxwell field. Subsequently this fact found reflection in the pseudotensor characteristic of a gravitational field, τ_p^l, introduced by Einstein. Later Schrödinger, 1918, demonstrated that if the coordinate system is chosen appropriately, all the components of the gravitational-field energy-momentum pseudotensor, τ_p^l, outside a ball vanish. In this connection Einstein, 1918a, noted:

> As for the ideas of Schrödinger, their persuasiveness lies in the analogy with electrodynamics, where the stresses and energy density of any field are nonzero. However, I cannot find any reason why the situation must be the same for gravitational fields. A gravitational field can be specified without introducing stresses and an energy density.

This clearly shows that *Einstein consciously departed from the concept of a gravitational field as being a Faraday-Maxwell physical field, a material substratum that can never be destroyed by the choice of reference frame.*

Since in GR there is no concept of a gravitational-field energy-momentum tensor density, there is no way of introducing in GR the law of conservation of energy-momentum of matter and gravitational field taken together. It was Hilbert, 1917, who first underlined this fact:

> I declare that ... for the general theory of relativity, that is, in the case of general invariance of the Hamiltonian function, there are generally no energy equations that ... correspond to energy equations in orthogonal-invariant theories. I could even note this fact as being a characteristic feature of the theory.

Unfortunately, this statement of Hilbert's was, apparently, not understood by his contemporaries, since neither Einstein nor other physicists realized at the time that GR in principle cannot have laws of conservation of energy-momentum and angular momentum of matter and gravitational field taken together. Even today there are scientists who do not understand this, while others do understand it but interpret it as the most important step made by GR, a step that overthrew the concept of energy. Rejection of the concept of the gravitational-field energy-momentum density leads to a situation in GR in which the gravitational-field energy cannot be localized. But the absence of any localization of field energy and the absence of conservation laws lead to the absence of the concepts of gravitational

waves and gravitational-wave flux, which means that the propagation of gravitational energy in space from one object to another is impossible in GR.

Another important aspect of GR is the fact that Einstein identified the metric of the Riemann space-time with the gravitational field. This too deprives the gravitational field within the GR framework of all the properties that a Faraday-Maxwell physical field has. The energy-momentum tensor of a physical field generated by a source cannot drop off, as we move away from the source, more slowly than $1/r^2$, since otherwise, if we took the integral of the law of energy-momentum conservation over the volume of a sphere surrounding the source and sent the radius of the sphere to infinity, we would arrive at a physically meaningless result, namely, that a source of finite dimensions carries an infinite amount of energy.

It has been established that the asymptotic behavior of metric coefficients depends on the choice of the three-dimensional spatial coordinates, and, consequently, this behavior can be made arbitrary. But this means that the metric of the Riemann space-time is not the physical gravitational field. Hence, to retain the concept of a gravitational field as being a Faraday-Maxwell physical field we must completely renounce its identity with the metric tensor.

In constructing the relativistic theory of gravitation, RTG, we will base our reasoning entirely on the special theory of relativity, which we call simply the relativity theory because physically there can be no other relativity. Although the name "general theory of relativity" does exist, it refers only to gravitation and not to some sort of general relativity. Long ago Fock, 1939, 1959, clarified this.

Now we briefly discuss the essence of the theory of relativity, touching especially on how Einstein interpreted the theory. This will not only be of historical interest; chiefly, it will give the reader a deeper understanding of the starting point of Einstein's reasoning which led to the creation of GR. Minkowski, following Poincaré's reasoning, developed the idea of the pseudo-Euclidean geometry of four-dimensional space-time. The line element in this geometry has the form

$$ds^2 = c^2\,dt^2 - dx^2 - dy^2 - dz^2.$$

Poincaré was the first to introduce such a quantity; he demonstrated that ds^2 is invariant under Lorentz transformations. He was also the first to introduce the concept of a Lorentz group and the idea of a four-dimensional space.

Even to this day many scientists believe that Minkowski provided a mathematical interpretation for the theory of relativity that formally simplified the theory. But there is more to it than this. In Logunov, 1985, it is demonstrated that the concept of a four-dimensional space-time developed by Poincaré and Minkowski makes it possible to extend the theory of relativity from inertial reference frames to accelerated frames. In an arbitrary accelerated reference frame the line element ds^2 has the form

$$ds^2 = \gamma_{ik}(x)\,dx^i dx^k,$$

where $\gamma_{ik}(x)$ is the metric tensor of the Minkowski space-time.

For the Minkowski space-time, in view of the existence of ten Killing vectors, there are always transformations

$$x^i = f^i(x')$$

that do not change the metric coefficients, that is,

$$ds^2 = \gamma_{ik}(x')\,dx'^i dx'^k.$$

It is this that enables us to generalize the Poincaré relativity principle (Poincaré, 1904, 1905) and formulate the generalized relativity principle thus (Logunov, 1985):

> No matter what physical reference frame we take (inertial or noninertial), there always exists an infinite number of other reference frames in which all physical phenomena (including gravitational phenomena) occur in the same manner as in the initial reference frame, so that we do not have and cannot have any experimental means to distinguish in which of this infinite number of reference frames we are positioned.

Thus, noninertial reference frames occupy an equal status with inertial reference frames in the theory of relativity. It is this fundamental fact that was not clear to Einstein (see Einstein and Grossmann, 1913):

> In the ordinary theory of relativity only linear orthogonal transformations are admitted.

Also (see Einstein, 1913),

> In the initial theory of relativity, the independence of the physical equations of the special choice of reference frame is based on the postulation of the fundamental invariant quantity
>
> $$ds^2 = \sum_i (dx_i)^2,$$
>
> whereas now we are speaking of constructing a theory (GR—*The authors*) in which a linear element of a more general nature plays the role of the fundamental invariant quantity
>
> $$ds^2 = \sum_{i,k} g_{ik}\, dx_i\, dx_k.$$

These passages show that at that time Einstein had yet to penetrate deep into the essence of the theory of relativity. In the special theory of relativity we are speaking not of the postulation of the line element in the form

$$ds^2 = \sum_i (dx_i)^2,$$

contrary to Einstein's belief, but of the pseudo-Euclidean geometry of space-time defined by the line element

$$ds^2 = \gamma_{ik}\, dx^i\, dx^k$$

with a metric tensor γ_{ik}, for which the Riemann-Cristoffel curvature tensor R_{inlm} vanishes. Hence, in the special theory of relativity the law of energy-momentum conservation can be written in the general covariant form, but this fact was not understood by Einstein. What has been said found its reflection in GR, in the construction of which Einstein was guided to a great extent by the elegant formal apparatus of Riemannian geometry and his idea of the equivalence of forces of inertia and gravity (the principle of equivalence).

According to the ideology of GR, the relativity principle cannot be applied to gravitational phenomena. It was on this central idea that Einstein and Hilbert, in creating GR almost 70 years ago, departed basically from the special theory of relativity, which in turn led to a rejection of the laws of conservation of energy-momentum and angular momentum, to the emergence of nonphysical ideas concerning the nonlocalization of gravitational energy, and to many other aspects not related to gravitation. These two great scientists abandoned the wonderfully simple world of the Minkowski space-time, which possesses the maximum possible (ten-parameter) group of motions on the space-time, and entered the jungle of Riemannian geometry, which bogged down subsequent generations of physicists studying gravitation.

Thus, if we assume that GR is a meaningful theory, we must reject both the fundamental laws of conservation of energy-momentum of matter and gravitational field and the concept of a classical field. This, however, is too great a loss, and it would be very thoughtless to agree to this without proper experimental proof. So far there is not a single experimental fact that, directly or indirectly, challenges the validity of conservation laws in the macro- and micro-worlds. There is only one conclusion then: we must discard GR, giving it credit as a stage in the development of our ideas of gravitation.

In Denisov and Logunov, 1980a, 1980b, 1982b, 1982d, Logunov and Folomeshkin, 1977b, and Vlasov and Denisov, 1982, it is demonstrated that since GR does not, and cannot, have laws of conservation of the energy-momentum of matter and gravitational field taken together, the inertial mass as defined in Einstein's theory has no physical meaning, the gravitational-wave flux as defined in GR can always be destroyed by the proper selection of reference frame, and, hence, Einstein's quadrupole formula for gravitational waves is not a corollary of GR. Basically it does not follow from GR that a binary system loses energy in the form of gravitational waves. GR has no classical Newtonian limit and, consequently, does not satisfy one of the most fundamental principles of physics, the correspondence principle. This is what the absence in GR of energy-momentum conservation laws leads to if one rejects dogmatism and ponders on the essence of the problem and makes a detailed analysis.

All this points to the fact that GR is not a satisfactory physical theory. Hence, it is urgent to construct a classical theory of gravitation that will satisfy all the demands made of a physical theory.

At the base of the suggested relativistic theory of gravitation (see Logunov, 1986, Logunov and Mestvirishvili, 1984, 1985a, 1985b, 1986b, Vlasov and Logunov, 1984, and Vlasov, Logunov, and Mestvirishvili, 1984), which completes the development of the ideas proposed in Denisov and Logunov, 1982d, we place the following physical requirements (see Logunov, 1986):

(a) The Minkowski space-time (x^m), that is, space-time equipped with pseudo-Euclidean geometry, is a fundamental space that incorporates all physical fields, including the gravitational. This statement is general because it is necessary and sufficient for the validity of the laws of conservation of energy-momentum and angular momentum for matter and gravitational field taken together. In other words, the Minkowski space-time reflects the dynamical properties common for all types of matter. This guarantees the existence of universal characteristics for all forms of matter and gravitational field. Discussing the structure of the geometry of real space-time, Einstein, 1921, noted:

> ...the question whether this continuum has a Euclidean, Riemannian, or any other structure is a question of physics proper which must be answered by experience, and not a question of a convention to be chosen on grounds of mere expediency.

Basically, of course, this statement of Einstein's is completely correct. But the essence of the matter is much deeper. The main thing here is to understand what physical properties of matter determine the geometry of space-time. Indeed, let us assume that if we determine the physical geometry on the basis of studies of the propagation of light and the movements of test bodies, we will establish the Riemannian structure of the geometry of space-time. But does this mean that this geometry must be placed at the base of the theory? No, it does not, because assuming this would mean rejecting automatically the fundamental laws of conservation of energy-momentum and angular momentum, since this geometry does not possess the maximum group of motions on space-time. And all this happened in GR.

Thus, once we have discovered on the basis of experiments involving the propagation of light and the movements of test bodies that Riemannian geometry is valid, we must not hasten to draw conclusions about the structure of the geometry of space-time that must be laid at the base of physics. We must first establish whether these experimental facts are primary and universal or of secondary origin and partial interest. *In establishing the structure of the geometry of the physical space-time we must proceed not from the nature of light propagation and test-body movements but from the most general dynamical properties of matter, the conservation laws, since it is not the particular physical manifestations of the motion of matter that determine the structure of the physical geometry lying at the base of physics but the general universal dynamical properties of matter.*

In our theory, RTG, the physical geometry of space-time is determined not on the basis of studies of the propagation of light and the movements of test bodies but on the basis of general dynamical properties of matter, the conservation laws, which are not only of fundamental importance but can be verified experimentally. Requirement (a) sets RTG entirely apart from the general theory of relativity.

(b) A gravitational field is described via a symmetric second-rank tensor Φ^{mn} and constitutes a real physical field characterized by an energy-momentum density, a zero rest mass, and spin states 2 and 0. This aspect also basically distinguishes RTG from GR.

(c) We introduce the geometrization principle, according to which the interaction of a gravitational field with matter is achieved, in view of the universality of this interaction, by "adding" the gravitational field Φ^{mn} to the metric tensor γ^{mn} of the Minkowski space-time in the Lagrangian density of matter according to the following rule:

$$L_{\mathrm{M}}(\tilde{\gamma}^{mn},\ \Phi_A) \rightarrow L_{\mathrm{M}}(\tilde{g}^{mn},\ \Phi_A),$$

where

$$\tilde{g}^{mn} = \sqrt{-g}\, g^{mn} = \sqrt{-\gamma}\, \gamma^{mn} + \sqrt{-\gamma}\, \Phi^{mn} \equiv \tilde{\gamma}^{mn} + \tilde{\Phi}^{mn},$$

and Φ_A are the material fields. By matter we mean all of its forms except gravitational fields. According to the geometrization principle, motion of matter under the action of a gravitational field Φ^{mn} in the Minkowski space-time with a metric γ^{mn} is equivalent to motion in an effective Riemann space-time with a metric g^{mn}. The metric tensor γ^{mn} of the Minkowski space-time and the gravitational-field tensor Φ^{mn} in this space-time are primary concepts, while the Riemann space-time and its metric g^{mn} are secondary concepts, owing their origin to the gravitational field and its universal action on matter through Φ_A. The effective Riemann space-time is literally of field origin, thanks to the presence of the gravitational field. Einstein was the first to suggest that the space-time is Riemannian rather than pseudo-Euclidean. He identified gravitation with the metric tensor of the Riemann space-time. But this line of reasoning as much as led to rejection of the gravitational field as a physical field possessing an energy-momentum density and to the loss of fundamental conservation laws. The geometrization principle, based on the notions of the Minkowski space-time and a physical gravitational field, introduces the concept of an effective Riemann space-time, and in this Einstein's idea of a Riemannian geometry finds its indirect reflection.

According to the RTG ideology, since the Minkowski space-time (x^m) forms its base, there are standard temporal and spatial scales that do not explicitly depend on the gravitational interaction. In view of the geometrization principle, the entire

dependence of the line element in the effective Riemann space-time,

$$ds^2 = g_{ik}(x)\, dx^i dx^k,$$

on the gravitational field lies in the metric coefficients $g_{ik}(x)$. Transition to any other coordinates in RTG, say, to the proper coordinates, will result in a situation in which the proper space-time variables will depend both on the coordinates x^m in the Minkowski space-time and on the gravitational constant G. Hence, proper time and spatial characteristics will depend on the gravitational field. It is only in RTG that one can completely determine the effect of a gravitational field on the passage of proper time and on the variation of the distance between points.

(d) The scalar Lagrangian density of a gravitational field is a bilinear form of the first covariant derivatives, $D_p\tilde{g}^{mn}$, with respect to the Minkowski metric. Basically there is no way to construct a scalar Lagrangian density of such a form in GR.

Using the concept of the Minkowski space-time and the geometrization principle as a basis, we can write the Lagrangian density in the following form:

$$L = L_g(\tilde{\gamma}^{ik}, \tilde{\Phi}^{ik}) + L_M(\tilde{g}^{ik}, \Phi_A). \tag{0.1}$$

In our theory the gravitational-field Lagrangian density L_g depends on the metric tensor γ^{ik} and the gravitational field Φ^{ik}. Hence, this theory differs fundamentally from GR, where the Lagrangian density depends only on the metric tensor g^{ik} of the Riemann space-time. Thus, the gravitational-field Lagrangian density in our theory is not fully geometrized, whereas in GR it is. As will be demonstrated later, the notion of a gravitational field possessing an energy-momentum density and spin states 2 and 0 combined with the geometrization principle provides the possibility of constructing an unambiguous relativistic theory of gravitation. Such a theory changes the stereotype of space-time developed under the influence of GR and in spirit agrees with the modern theories in elementary particle physics. It implies that Einstein's general relativity principle is devoid of any physical meaning or content (Fock, 1939, 1959). In the exposition of a number of problems we follow Denisov and Logunov, 1982d.

Chapter 1. Critical Remarks Concerning the Equivalence Principle

In the introduction we specified the logical prerequisites that must necessarily lead (and, indeed, do) to a number of difficulties in GR. Earlier these aspects were discussed in detail in Denisov and Logunov, 1980a, 1980b, 1982b, 1982d, Fock, 1939, 1959, Hilbert, 1917, Logunov and Folomeshkin, 1977b, Schrödinger, 1918, and Vlasov and Denisov, 1982. In this chapter we intend to consider some of these aspects, following Denisov and Logunov, 1980a, 1980b, 1982b, 1982d, Logunov and Folomeshkin, 1977b, and Vlasov and Denisov, 1982, and show that GR is incapable of resolving these difficulties.

We start with a discussion of the equivalence principle. To the present day in the literature there is no unity of opinion concerning the meaning of the equivalence principle and its role in the general theory of relativity. Some consider it central to the structure of GR, while others note its limited nature. In the first stage of creating his theory, Einstein used as a leading idea the formal analogy between a field of forces of inertia and a gravitational field. Indeed, these fields demonstrate much in common in their action on the mechanical motion of objects: the motion of objects under the action of a gravitational field is indistinguishable from their motion in an appropriately chosen noninertial reference frame; in both fields the acceleration of objects does not depend on their mass or composition. This gave Einstein the grounds to state that the gravitational mass of an object is exactly equal to the object's inertial mass and led him to the formulation of the equivalence principle (Einstein and Grossmann, 1913):

> The theory described here originates from the conviction that the proportionality between the inertial and the gravitational mass of a body is an exact law of nature that must be expressed as a foundation principle of theoretical physics. We tried to reflect this conviction in a number of earlier papers, in which an attempt was made to reduce the *gravitational mass* to the *inertial mass*; this aspiration led us to the hypothesis that physically a gravitational field (homogeneous in an infinitely small volume) can be completely replaced with an accelerated reference frame. Graphically this hypothesis can be formulated as follows: An observer enclosed in an elevator has no way to decide whether the elevator is at rest in a static gravitational field or whether the elevator is located in gravitation-free space in an accelerated motion that is maintained by forces acting on the elevator (equivalence hypothesis).

Thus, from Einstein's point of view, the only difference between fields of forces of inertia and gravitational fields consists in the different external sources generating these fields: the first are due to the noninertiality of the reference frame used by the observer and the second are generated by material objects. However, as Einstein believed, these fields have an equivalent effect on all physical processes and, therefore, in other respects are indistinguishable. This statement, in turn, created the illusion of the possibility of excluding the effect of a gravitational field on all physical phenomena via an appropriate transformation of the space-time coordinates, by analogy with the destruction of fields of forces of inertia.

In this respect the following passage from Pauli, 1958 (p. 145), is characteristic:

> Originally, the principle of equivalence had only been postulated for homogeneous gravitational fields. For the general case, it can be formulated in the following way: For every infinitely small world region (i.e. a world region which is so small that the space-and-time-variation of gravity can be neglected in it) there always exists a coordinate system K_0 (X_1, X_2, X_3, X_4) in which gravitation has no influence either on the motion of particles or any other physical processes. In short, in an infinitely small world region every gravitational field can be transformed away ... .

A similar statement can also be found in Einstein, 1925:

> ...for any infinitely small world-region the coordinates can always be chosen in such a way that the gravitational field in that world-region vanishes. Then we may assume that in such an infinitely small world-region the special theory of relativity is valid. In this way the general theory of relativity is related to the special theory of relativity and the results of the latter are transformed to the former.

Later these erroneous statements found their way into a number of textbooks practically without alterations. However, forces of inertia and gravitational forces are entirely different in their origin, since for the first the curvature tensor is identically zero while for the second it is nonzero. Consequently, the effect of the first on all physical phenomena can be nullified in the entire space (globally) by transferring to an inertial reference frame, while the effect of gravitational forces can be destroyed only in local regions of space and not for all physical processes but only for the simplest, those in whose equations the space-time curvature is not present.

Hence, on the one hand, the principle of equivalence is invalid for processes involving particles with higher spins because the equations for the particles contain the curvature tensor explicitly; on the other, the principle cannot be applied to extended objects (of sufficiently large dimensions) since in this case the deviation of the geodesics corresponding to the edge points of the object manifests itself. Since the curvature tensor enters into the deviation equation, forces of inertia and gravitational forces are nonequivalent for mechanical movements of an extended object, too.

The main credit for elucidating these aspects must be given to Eddington, 1923 (pp. 40 and 41), who pointed out:

> The Principle of Equivalence has played a great part as a guide in the original building up of the generalized relativity theory; but now that we have reached the new view of the nature of the world it has become less necessary .... It is essentially an hypothesis to be tested by experiment as opportunity offers. Moreover it is to be regarded as a suggestion, rather than a dogma admitting of no exceptions. It is likely that some of the phenomena will be determined by the comparatively simple equations in which the components of curvature of the world do not appear; such equations will be the same for a curved region as for a flat region. It is to these that the Principle of Equivalence applies.

One cannot, however, assert that descriptions of physical phenomena in a gravitational field and in a noninertial reference frame of the pseudo-Euclidean space-time are fully equivalent, since (as put by Eddington):

> ...there are more complex phenomena governed by equations in which the curvatures of the world are involved; terms containing these curvatures will vanish in the equations summarising experiments made in a flat region, and would have to be reinstated in passing to the general equations. Clearly there must be some phenomena of this kind which discriminate between a flat world and a curved world; otherwise we could have no knowledge of world-curvature. For these the Principle of Equivalence breaks down.

Thus, the equivalence principle, if understood as the possibility of excluding the gravitational field in an infinitesimal region, is not correct since there is no way in which we can exclude the curvature of space (if it is nonzero) by selecting an appropriate reference frame, even within a given accuracy. Moreover, grav-

itational fields and fields of forces of inertia do not have a similar effect on all physical processes.

True, it must be noted that subsequently Einstein reconsidered his viewpoint on the principle of equivalence and did not insist on the complete equivalence of fields of forces of inertia and gravitational fields, pointing out that the former (non-inertial reference frames) constitute a particular case of gravitational fields satisfying the Riemann condition $R^i_{nml} = 0$. He wrote (see Einstein, 1949):

> There is a special kind of space whose physical structure (field) can be presumed as precisely known on the basis of the special theory of relativity. This is empty space without electromagnetic field and without matter. It is completely determined by its metric property: Let dx_0, dy_0, dz_0, and dt_0 be the coordinate differences of two infinitesimally near points (events); then
>
> (1) $$ds^2 = dx_0^2 + dy_0^2 + dz_0^2 - c^2\,dt_0^2$$
>
> is a measurable quantity that is independent of the special choice of the inertial system. If one introduces in this space the new coordinates x_1, x_2, x_3, and x_4 through a general transformation of coordinates, then the quantity ds^2 for the same pair of points has an expression of the form
>
> (2) $$ds^2 = \sum g_{ik} dx^i dx^k$$
>
> (summed for i and k from 1 to 4), where $g_{ik} = g_{ki}$. The g_{ik} that form a symmetric tensor and are continuous functions of $x_1, \ldots, x_4$ then describe according to the principle of equivalence a gravitational field of a special kind, namely, one that can be retransformed to the form (1).
>
> From Riemann's investigations on metric spaces the mathematical properties of this g_{ik} field can be given exactly (Riemann condition). What is sought, however, is the equations satisfied by general gravitational fields. It is natural to assume that they, too, can be described as tensor fields of the type g_{ik}, which in general do not admit a transformation to the form (1), that is, which do not satisfy the Riemann condition, but are weaker conditions, which, just as the Riemann condition, are independent of the choice of coordinates (that is, are generally invariant). A simple formal consideration leads to weaker conditions that are closely connected with the Riemann condition. These conditions are the very equations of the pure gravitational field (on the outside of matter and in the absence of an electromagnetic field).

Thus, *Einstein altered the physical meaning of the equivalence principle, although this fact apparently remained unnoticed by many.*

While creating the general theory of relativity, Einstein was totally guided by the principle of equivalence in its initial wording, which therefore played a heuristic role in constructing the theory (Einstein and Grossmann, 1914):

> The entire theory originated on the basis of the conviction that in a gravitational field all physical processes occur in the same manner as in the absence of gravitational field but in an appropriately accelerated (three-dimensional) system of coordinates (equivalence hypothesis).

Since in those days, thanks to Minkowski's discovery, it was known that to different reference frames there correspond different (generally off-diagonal) metrics of space-time, Einstein and Grossmann, 1913, concluded that *the metric tensor of the Riemann space-time must be taken as the field variable for the gravitational field* and that this tensor is determined by the distribution and motion of matter. In this way there emerged the idea of a link between matter and the geometry of space-time.

Proceeding from these assumptions, Einstein and Grossmann purely intuitively tried to establish the form of the equations linking the components of the metric tensor of the Riemann space-time with the energy-momentum tensor for matter. After numerous unsuccessful attempts such equations were found by Einstein at the end of 1915. Since somewhat earlier Hilbert, 1915, arrived at the same equations

(his reasoning was based on variational principles), we will call these equations the Hilbert-Einstein equations.

It must be noted that the metric tensor of the Riemann space-time cannot serve as a characteristic of the gravitational field because its asymptotic behavior depends on the choice of the three-dimensional (spatial) system of coordinates.

Chapter 2. Energy-Momentum Pseudotensors of the Gravitational Field in GR

Einstein believed that in GR the gravitational field together with matter must obey a conservation law of some kind (Einstein, 1914):

> ...it goes without saying that we must require that matter and gravitational field taken together satisfy energy and momentum conservation laws.

In his opinion, this problem had been fully solved on the basis of "conservation laws" that used the energy-momentum pseudotensor as the energy-momentum characteristic of the gravitational field. The common line of reasoning that leads to such "conservation laws" goes as follows (Landau and Lifshitz, 1975). If the Hilbert-Einstein equations are written as

$$-\frac{c^4}{8\pi G}\, g\left[R^{ik}-\frac{1}{2}\, g^{ik}R\right]=-gT^{ik}, \tag{2.1}$$

where $g = \det g_{ik}$, R^{ik} is the Ricci tensor, and T^{ik} the energy-momentum tensor for matter, then the left-hand side can be represented as a sum of two noncovariant quantities:

$$-\frac{c^4}{8\pi G}\, g\left[R^{ik}-\frac{1}{2}\, g^{ik}R\right]=\frac{\partial}{\partial x^l}\, h^{ikl}+g\tau^{ik}, \tag{2.2}$$

where $\tau^{ik} = \tau^{ki}$ is the gravitational-field energy-momentum pseudotensor, and $h^{ikl} = -h^{ilk}$ the spin pseudotensor. This transforms the Hilbert-Einstein equations (2.1) into an equivalent form

$$-g\,[T^{ik}+\tau^{ik}]=\frac{\partial}{\partial x^l}\, h^{ikl}. \tag{2.3}$$

In view of the obvious fact that

$$\frac{\partial^2}{\partial x^k\,\partial x^l}\, h^{ikl}=0, \tag{2.4}$$

the Hilbert-Einstein equations (2.3) yield the following differential conservation law:

$$\frac{\partial}{\partial x^k}\,[-g\,(T^{ik}+\tau^{ik})]=0, \tag{2.5}$$

which formally is similar to the conservation law for energy-momentum in electrodynamics. According to GR, this law is valid for any choice of coordinates x^k, for one thing, spherical coordinates $(t,\ r,\ \theta,\ \varphi)$. But in the latter case (2.5) will always lead to physically meaningless results. Thus, Eqs. (2.1) in arbitrary coordinates always lead to (2.5), which has no physical meaning.

In accordance with this analogy, the gravitational energy "flux" through the elemental surface area dS_α is defined in GR thus:

$$dI = c\,(-g)\,\tau^{0\alpha}\, dS_\alpha.$$

Taking a sphere of radius r as the surface of integration ($dS_\alpha = -r^2 n_\alpha d\Omega$), we arrive at the formula for the "intensity" of gravitational energy per unit solid angle:

$$\frac{dI}{d\Omega} = -cr^2(-g)\tau^{0\alpha}n_\alpha. \tag{2.6}$$

Formula (2.5) is also used in GR to derive integral "conservation laws for the energy-momentum" of matter and gravitational field taken together. Here usually (see Einstein, 1918b, and Landau and Lifshitz, 1975) (2.5) is integrated over a definite volume and then it is assumed that the matter fluxes through the surface confining the volume of integration are zero. The result is

$$\frac{d}{dt}\int(-g)[T^{0i}+\tau^{0i}]\,dV = -\oint(-g)\tau^{\alpha i}\,dS_\alpha. \tag{2.7}$$

Einstein, 1918b, assumed that the right-hand side of (2.7) at $i = 0$ is "for certain the loss of energy by the material system" and, hence,

$$-\frac{dE}{dt} = \oint(-g)\tau^{0\alpha}\,dS_\alpha. \tag{2.8}$$

In the absence of gravitational-field "energy-momentum fluxes" through the surface confining the integration volume, Eq. (2.7) yields the following law of "energy-momentum" conservation in the system:

$$P^i = \frac{1}{c}\int(-g)[T^{0i}+\tau^{0i}]\,dV = \text{const}. \tag{2.9}$$

By means of the Hilbert-Einstein equations (2.3), the law (2.9) can be rewritten as follows:

$$P^i = \frac{1}{c}\oint h^{0i\alpha}\,dS_\alpha = \text{const}. \tag{2.10}$$

Einstein, 1918c, believed that the four quantities P^i constitute the energy ($i = 0$) and the momentum ($i = 1, 2, 3$) of a physical system. It is usually stated in this connection (see Landau and Lifshitz, 1975, pp. 283-284) that

> The quantities P^i (the four-momentum of field plus matter) have a completely definite meaning and are independent of the choice of reference system to just the extent that is necessary on the basis of physical considerations.

We will show, however, that this statement is erroneous (see Chapter 3).

On the basis of such a definition of the "energy-momentum" of a system consisting of matter and gravitational field, the following concept of inertial mass m_1 of the system is introduced in GR:

$$m_1 = \frac{1}{c}P^0 = \frac{1}{c^2}\int(-g)[T^{00}+\tau^{00}]\,dV = \frac{1}{c^2}\oint h^{00\alpha}\,dS_\alpha. \tag{2.11}$$

Expressions similar to (2.5)-(2.11) can also be obtained if the Hilbert-Einstein equations are written in terms of mixed components:

$$\sqrt{-g}\,[T_i^m + \tau_i^n] = \partial_m \sigma_i^{mn}.$$

The choice of the gravitational-field energy-momentum pseudotensors depended to a great extent on the preference of the different authors and, as a rule, was carried out on the basis of secondary properties. For instance, if we take h^{ikl} in the form

$$h^{ikl} = \frac{c^4}{16\pi G}\frac{\partial}{\partial x^m}[-g(g^{ik}g^{ml} - g^{il}g^{mk})], \tag{2.12}$$

we arrive at the Landau-Lifshitz symmetric pseudotensor, which contains only the first derivatives of the metric tensor:

$$\tau^{ik} = \frac{c^4}{16\pi G}\{(2\Gamma^n_{ml}\Gamma^p_{np} - \Gamma^n_{lp}\Gamma^p_{mn} - \Gamma^n_{nl}\Gamma^p_{mp})(g^{il}g^{mk} - g^{ik}g^{ml})$$
$$+ g^{il}g^{mn}(\Gamma^k_{lp}\Gamma^p_{mn} + \Gamma^k_{mn}\Gamma^p_{lp} - \Gamma^k_{np}\Gamma^p_{ml} - \Gamma^k_{ml}\Gamma^p_{np})$$
$$+ g^{kl}g^{mn}(\Gamma^i_{pl}\Gamma^p_{mn} + \Gamma^i_{mn}\Gamma^p_{pl} - \Gamma^i_{np}\Gamma^p_{ml} - \Gamma^i_{ml}\Gamma^p_{np})$$
$$+ g^{ml}g^{np}(\Gamma^i_{nl}\Gamma^k_{mp} - \Gamma^i_{ml}\Gamma^k_{np})\}, \tag{2.13}$$

where

$$\Gamma^k_{mn} = \frac{1}{2}g^{kp}(\partial_m g_{pn} + \partial_n g_{pm} - \partial_p g_{mn}).$$

If we assume that

$$\sigma^{ni}_k = \frac{c^4 g_{km}}{6\pi G\sqrt{-g}}\frac{\partial}{\partial x^l}[-g(g^{mi}g^{nl} - g^{mn}g^{il})], \tag{2.14}$$

we arrive at Einstein's pseudotensor

$$\tau^i_k = \frac{c^4\sqrt{-g}}{16\pi G}\{-2\Gamma^i_{ml}\Gamma^l_{kp}g^{mp} + \Gamma^l_{ml}\Gamma^m_{kp}g^{ip} + \Gamma^l_{lm}\Gamma^i_{kp}g^{mp} + \Gamma^l_{kl}\Gamma^i_{mp}g^{mp}$$
$$- \Gamma^l_{kl}\Gamma^p_{mp}g^{mi} - \delta^i_k[g^{mp}\Gamma^l_{mp}\Gamma^n_{ln} - g^{nl}\Gamma^p_{ml}\Gamma^m_{pn}]\}, \tag{2.15}$$

which coincides with the canonical energy-momentum (pseudo) tensor obtained from the noncovariant gravitational-field Lagrangian density

$$L_g = \sqrt{-g}\,g^{li}[\Gamma^n_{pi}\Gamma^p_{nl} - \Gamma^n_{li}\Gamma^p_{np}].$$

At

$$\sigma^{ni}_k = \frac{c^4\sqrt{-g}}{16\pi G}g^{mi}g^{nl}[\partial_l g_{km} - \partial_m g_{kl}] \tag{2.16}$$

we have Lorentz's pseudotensor

$$\tau^i_k = \frac{c^4\sqrt{-g}}{16\pi G}[\partial_k\Gamma^l_{pl}g^{pi} - \partial_k\Gamma^i_{mp}g^{mp} - \delta^i_k R], \tag{2.17}$$

which coincides with the canonical energy-momentum (pseudo) tensor obtained via the noncovariant infinitesimal-displacement method from the covariant gravitational-field Lagrangian density $L_g = \sqrt{-g}R$.

Let us now investigate the "energy-momentum" quantities introduced in Einstein's theory using the definition of "inertial mass" of a spherically symmetric source. For the sake of definiteness we carry out all calculations within the framework of the Landau-Lifshitz symmetric pseudotensor.

Chapter 3. Inertial Mass in GR

Einstein considered the equality of inertial and gravitational mass of an object as an exact law of nature, a law that must find its reflection in his theory. At present it is taken for granted in GR that the gravitational mass of a system consisting of matter and gravitational field is equal to the inertial mass of the system. Such statements are contained in the works of Einstein, 1918c, Tolman, 1934, and Weyl, 1923. Subsequently, the "proof" of this statement with various altera-

tions was carried out by other authors (see Landau and Lifshitz, 1975, Misner, Thorne, and Wheeler, 1973, and Møller, 1952). However, the statement is erroneous. Following Denisov and Logunov, 1982b, we will now prove this.

The gravitational mass M of an arbitrary physical system that is as a whole at rest with respect to a Galilean (at infinity) Schwarzschild system of coordinates was defined by Einstein, 1918c, as the quantity that is the factor of the term $-2G/c^2r$ in the asymptotic expression (as $r \to \infty$) for the component g_{00} of the metric tensor of the Riemann space-time:

$$g_{00} = 1 - \frac{2G}{c^2 r} M.$$

A somewhat different definition was given by Tolman, 1934:

$$M = \frac{c^2}{4\pi G} \int R_0^0 \sqrt{-g}\, dV. \tag{3.1}$$

These definitions imply directly that gravitational mass is invariant under transformations of three-dimensional coordinates, since both the component R_0^0 of the Ricci tensor and the component g_{00} of the metric tensor are transformed like a scalar.

For the case of a static spherically symmetric source these definitions are equivalent. We wish to demonstrate that they remain so for any static system. To this end we write R_0^0 in the form

$$R_0^0 = g^{0i} \left[\frac{\partial}{\partial x^l} \Gamma_{0i}^l - \frac{\partial}{\partial x^0} \Gamma_{pi}^p + \Gamma_{0i}^n \Gamma_{np}^p - \Gamma_{pi}^n \Gamma_{n0}^p \right].$$

Identity transformations yield

$$\begin{aligned} R_0^0 = {} & \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\alpha} [\sqrt{-g}\, g^{0n} \Gamma_{0n}^\alpha] - g^{0i} \frac{\partial}{\partial x^0} \Gamma_{ni}^n \\ & - \frac{1}{2} \Gamma_{ni}^0 \frac{\partial g^{ni}}{\partial x^0} + \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^0} [\sqrt{-g}\, g^{0n} \Gamma_{0n}^0]. \end{aligned} \tag{3.2}$$

Since for static systems the last three terms on the right-hand side of (3.2) can be ignored, (3.1) yields

$$M = \frac{c^2}{4\pi G} \oint dS_\alpha \sqrt{-g}\, g^{0n} \Gamma_{0n}^\alpha. \tag{3.3}$$

Since quite far from the static system its metric can be described, with a given accuracy, by the Schwarzschild metric, (3.3) assumes the form

$$M = -\frac{c^2}{8\pi G} \lim_{r\to\infty} \oint dS_\alpha g^{00} \sqrt{-g} \frac{\partial}{\partial x^\alpha} g_{00}. \tag{3.4}$$

The integrand in (3.1) is a scalar for all transformations of the three-dimensional coordinate system, which means that the gravitational mass M is independent of the choice of coordinates. In Schwarzschild coordinates, (3.4) assumes the form

$$M = \frac{c^2}{2G} \lim_{r\to\infty} \left(r^2 \frac{\partial g_{00}}{\partial r} \right) = \frac{c^2}{2G} \lim_{r\to\infty} \left[r^2 \frac{\partial}{\partial r} \left(1 - \frac{2G}{c^2 r} M \right) \right].$$

Thus, the gravitational mass of any static system, according to Tolman's definition, is the factor of $-2G/c^2r$ in the asymptotic expression for the component g_{00} of the metric tensor of the Riemann space-time. Hence, for static systems the definitions of gravitational mass given by Einstein and Tolman coincide.

Einstein closely linked the concept of the inertial mass of a physical system in GR with the idea of the energy of the system (Einstein, 1918c):

> ...the quantity that we have interpreted as energy plays the role of inertial mass, in accordance with the special theory of relativity.

Since Einstein suggested calculating the energy of a system within the GR framework with the aid of energy-momentum pseudotensors, the inertial mass is calculated on the basis of (2.11).

We now define in accordance with (2.11) the inertial mass of a spherically symmetric source of gravitational field and study how inertial mass transforms under coordinate transformations. In isotropic Cartesian coordinates the metric of the Riemann space-time has the form

$$g_{00}=\frac{(1-r_g/4r)^2}{(1+r_g/4r)^2}, \quad g_{\alpha\beta}=\gamma_{\alpha\beta}\,(1+r_g/4r)^4, \tag{3.5}$$

where $r_g = 2GM/c^2$. These coordinates are asymptotically Galilean, since the following estimates hold true as $r \to \infty$:

$$g_{00}=1+O(1/r), \quad g_{\alpha\beta}=\gamma_{\alpha\beta}\,[1+O(1/r)]. \tag{3.6}$$

If we employ the covariant components of metric (3.5), then (2.12) yields

$$h^{00\alpha}=-\frac{c^4}{16\pi G}\frac{\partial}{\partial x^\beta}\,[g_{11}g_{22}g_{33}g^{\alpha\beta}].$$

Substituting this into (2.10), allowing for the fact that

$$dS_\alpha=-\frac{x_\alpha}{r}\,r^2\sin\theta\,d\theta\,d\varphi, \tag{3.7}$$

and integrating over an infinitely distant surface, we obtain

$$P^0=\frac{c^3}{16\pi G}\lim_{r\to\infty} r^2\int\frac{x_\alpha}{r}\frac{\partial}{\partial x^\beta}\,[-g_{11}g_{22}g_{33}g^{\alpha\beta}]\sin\theta\,d\theta\,d\varphi. \tag{3.8}$$

Thus, the component P^0 is independent of the component g_{00} of the metric tensor of the Riemann space-time. Combining (3.5), (3.8), and

$$\frac{\partial}{\partial x^\beta}f(r)=\frac{x^\beta}{r}\frac{\partial}{\partial r}f(r), \tag{3.9}$$

where $r^2 = -x_\alpha x^\alpha$, we arrive at the following expression for the component P^0 of "energy-momentum":

$$P^0=\frac{c^3 r_g}{2G}=Mc. \tag{3.10}$$

It was the fact that "inertial mass" coincides with gravitational mass that gave grounds for asserting that they are equal in GR, too. Landau and Lifshitz, 1975 (p. 334), wrote:

> ...$P^\alpha = 0$, $P^0 = Mc$, a result which was naturally to be expected. It is an expression of the equality of "gravitational" and "inertial" mass ("gravitational" mass is the mass that determines the gravitational field produced by the body, the same mass that appears in the metric tensor in a gravitational field, or, in particular, in Newton's law; "inertial" mass is the mass that determines the ratio of energy and momentum of the body; in particular, the rest energy of the body is equal to this mass multiplied by c^2).

However, this statement and similar statements made by Einstein, 1918c, and other authors (see Eddington, 1923, Misner, Thorne, and Wheeler, 1973, Møller,

1952, and Tolman, 1934) are erroneous. As can easily be demonstrated, the "energy" of a system and, hence, the "inertial mass" of the same system, (2.11), have no physical meaning because their magnitude depends on the choice of the three-dimensional coordinate system. Indeed, a basic requirement that any definition of inertial mass must satisfy is the independence of this quantity from the choice of the three-dimensional system of coordinates; this holds true for any physical theory. But in GR the definition (2.11) of "inertial mass" does not meet this requirement.

We will demonstrate, for instance, that in the case of a Schwarzschild solution the "inertial mass" (2.11) assumes an arbitrary value depending on the choice of the three-dimensional coordinate system. To this end we transfer from three-dimensional Cartesian coordinates x_{C}^{α} to other coordinates x_{N}^{α} linked to the previous coordinates by the following formula:

$$x_{\mathrm{C}}^{\alpha} = x_{\mathrm{N}}^{\alpha}[1 + f(r_{\mathrm{N}})], \tag{3.11}$$

where $r_{\mathrm{N}} = (x_{\mathrm{N}}^2 + y_{\mathrm{N}}^2 + z_{\mathrm{N}}^2)^{1/2}$, and $f(r_{\mathrm{N}})$ is an arbitrary nonsingular function obeying the conditions

$$f(r_{\mathrm{N}}) \geqslant 0, \quad \lim_{r_{\mathrm{N}} \to \infty} f(r_{\mathrm{N}}) = 0, \quad \lim_{r_{\mathrm{N}} \to \infty} r_{\mathrm{N}} \frac{\partial}{\partial r_{\mathrm{N}}} f(r_{\mathrm{N}}) = 0. \tag{3.12}$$

It is easy to see that transformation (3.11) corresponds to a change in the arithmetization of the points of the three-dimensional space along the radius, $r_{\mathrm{C}} = r_{\mathrm{N}}[1 + f(r_{\mathrm{N}})]$. For transformation (3.11) to have an inverse and be one-to-one, it is necessary and sufficient that

$$\frac{\partial r_{\mathrm{C}}}{\partial r_{\mathrm{N}}} = 1 + f + r_{\mathrm{N}} f' > 0,$$

where

$$f' = \frac{\partial}{\partial r_{\mathrm{N}}} f(r_{\mathrm{N}}).$$

Then the Jacobian of this transformation will also be nonzero:

$$J = \det\left[\frac{\partial x_{\mathrm{C}}}{\partial x_{\mathrm{N}}}\right] = (1+f)^2 \frac{\partial r_{\mathrm{C}}}{\partial r_{\mathrm{N}}} \neq 0.$$

Specifically, the function

$$f(r_{\mathrm{N}}) = \alpha^2 \left(\frac{8GM}{c^2 r_{\mathrm{N}}}\right)^{1/2} [1 - \exp(-\varepsilon^2 r_{\mathrm{N}})], \tag{3.13}$$

with α and ε arbitrary numbers not equal to zero, satisfies every one of the above requirements. If we allow for (3.13), we get

$$\frac{\partial r_{\mathrm{C}}}{\partial r_{\mathrm{N}}} = 1 + \alpha^2 \left(\frac{8GM}{c^2 r_{\mathrm{N}}}\right)^{1/2} \left[\frac{1}{2} + \left(\varepsilon^2 r_{\mathrm{N}} - \frac{1}{2}\right) \exp(-\varepsilon^2 r_{\mathrm{N}})\right] > 0,$$

which shows that r_{C} is a monotone function of r_{N}. It is easy to verify that $f(r_{\mathrm{N}})$ is a nonnegative nonsingular function in the entire space. The Jacobian of the transformation in this case is strictly greater than unity:

$$J = (1+f)^2 \frac{\partial r_{\mathrm{C}}}{\partial r_{\mathrm{N}}} > 1.$$

Therefore, transformation (3.11) with function $f(r_{\mathrm{N}})$ defined via (3.13) has an inverse transformation and is one-to-one.

Obviously, transformation (3.11) does not change the value of the gravitational mass (3.1). We will now calculate the value of the "inertial mass" (2.11) in terms of the new coordinates x_N^α. Applying the tensor transformation law to the metric tensor,

$$g_{ni}^{N} = \frac{\partial x_C^l}{\partial x_N^n} \frac{\partial x_C^m}{\partial x_N^i} g_{ml}^{C} [x_C(x_N)], \tag{3.14}$$

we can find the components of the Schwarzschild metric (3.5) in terms of the new coordinates. The result is

$$\begin{gathered} g_{00} = \left[1 - \frac{r_g}{4r_N(1+f)}\right]^2 \left[1 + \frac{r_g}{4r_N(1+f)}\right]^{-2}, \\ g_{\alpha\beta} = \left[1 + \frac{r_g}{4r_N(1+f)}\right]^4 \left\{\gamma_{\alpha\beta}(1+f)^2 - x_\alpha^N x_\beta^N \left[(f')^2 + \frac{2}{r_N} f'(1+f)\right]\right\}. \end{gathered} \tag{3.15}$$

The determinant of the metric tensor (3.15) is

$$g = -g_{00}\left[1 + \frac{r_g}{4r_N(1+f)}\right]^{12} (1+f)^4 [(1+f)^2 + r_N^2 (f')^2 + 2r_N f'(1+f)]. \tag{3.16}$$

It must be especially noted that metric (3.15) is asymptotically Galilean:

$$\lim_{r_N \to \infty} g_{00} = 1, \quad \lim_{r_N \to \infty} g_{\alpha\beta} = \gamma_{\alpha\beta}.$$

In the particular case where function f is specified by (3.13) and r_N is sent to infinity, the metric of the Riemann space-time has the following asymptotic behavior:

$$g_{00} \simeq 1 + O(1/r_N), \quad g_{\alpha\beta} = \gamma_{\alpha\beta}[1 + O(1/r_N^{1/2})]. \tag{3.17}$$

For the covariant components of metric (3.15) we have

$$g^{00} = g_{00}^{-1}, \quad g^{\alpha\beta} = \gamma^{\alpha\beta} A + x_N^\alpha x_N^\beta B, \tag{3.18}$$

where we have introduced the notation

$$A = (1+f)^{-2}\left[1 + \frac{r_g}{4r_N(1+f)}\right]^{-4},$$

$$B = \frac{r_N(f')^2 + 2f'(1+f)}{r_N\left[1 + \frac{r_g}{4r_N(1+f)}\right]^4 (1+f)^2 [(1+f)^2 + r_N^2 (f')^2 + 2r_N f'(1+f)]}.$$

Substituting (3.16) and (3.18) into (3.8), we obtain

$$\begin{aligned} P^0 = \frac{c^3}{16\pi G} \lim_{r_N \to \infty} r_N^2 \int \frac{x_\alpha^N}{r_N} \frac{\partial}{\partial x_N^\beta} \Big\{ & \gamma^{\alpha\beta}(1+f)^2 \left[1 + \frac{r_g}{4r_N(1+f)}\right]^8 \\ & \times [(1+f)^2 + r_N^2(f')^2 + 2r_N f'(1+f)] \\ & + \frac{x_N^\alpha x_N^\beta}{r_N^2} (1+f)^2 \left[1 + \frac{r_g}{4r_N(1+f)}\right]^8 \\ & \times [r_N^2(f')^2 + 2r_N f'(1+f)] \Big\} \, dV. \end{aligned}$$

In view of the validity of (3.9) this yields

$$P^0 = \frac{c^3}{2G} \lim_{r_N \to \infty} \left\{ r_N^3 (f')^2 (1+f)^2 \left[1 + \frac{r_g}{4r_N (1+f)} \right]^8 + r_g (1+f)^2 (1+f+r_N f') \left[1 + \frac{r_g}{4r_N (1+f)} \right]^7 \right\}. \quad (3.19)$$

Allowing for the asymptotic expression (3.12) for f, we finally obtain (see Møller, 1965)

$$P^0 = \frac{c^3}{2G} \lim_{r_N \to \infty} \{ r_g + r_N^3 (f')^2 \}. \quad (3.20)$$

Thus, the "inertial mass" depends essentially on the rate at which f' tends to zero as $r_N \to \infty$. Specifically, if we take function $f(r_N)$ in the form (3.13), from (3.20) we get

$$m_1 = M(1 + \alpha^4). \quad (3.21)$$

Hence for the "inertial mass" (2.11) of a system consisting of matter and gravitational field we can obtain in GR any fixed number $m_1 \geqslant M$ depending on the choice of the spatial coordinate system (because of the arbitrariness of α), whereas the gravitational mass M (3.1) of this system and, consequently, all three effects of GR remain unaltered. Note also that in the event of more complex transformations of the spatial coordinates that leave the metric asymptotically Galilean, the "inertial mass" (2.11) of the system can assume any fixed values, positive as well as negative.

Thus, we see that in GR the value of the "inertial mass", a concept introduced by Einstein and later used by many authors (e.g. see Eddington, 1923, Landau and Lifshitz, 1975, Misner, Thorne, and Wheeler, 1973, Møller, 1952, and Tolman, 1934), depends on the choice of the three-dimensional system of coordinates and, therefore, carries no physical meaning. Hence, *the statement that "inertial mass" is equal to gravitational mass in Einstein's theory has no physical meaning either.* The equality takes place in a narrow class of three-dimensional systems of coordinates, and since the "inertial mass" (2.11) and the gravitational mass (3.1) obey different transformation laws, a transition to other three-dimensional systems of coordinates results in a violation of this equality.

More than that, such a definition of "inertial mass" in GR does not obey the principle of correspondence to Newton's theory. Indeed, since the "inertial mass" m_i in Einstein's theory depends on the choice of the three-dimensional system of coordinates, its expression in the general case of an arbitrary three-dimensional coordinate system does not transform into the appropriate expression in Newton's theory, in which the "inertial mass" does not depend on such a choice. Thus, *GR contains no classical Newtonian limit and, hence, does not satisfy the correspondence principle. This implies that GR is not only logically contradictory from the viewpoint of physics but also directly contradicts the experimental data on the equality of the inertial and the active gravitational mass.*

Then why were the aforesaid and the necessary conclusions not arrived at earlier? Apparently the answer is that Einstein focused on the problem of energy-momentum in GR and after studying it assumed he had succeeded in finding a solution that was definite to the same extent as in classical mechanics. Somewhat later Klein, 1918, mathematically substantiated Einstein's ideas. Einstein's conclusions concerning the energy-momentum of a system are repeated almost without vari-

ation to this day (e.g. see Landau and Lifshitz, 1975). The studies of these outstanding scientists created the belief that in GR the problem of energy-momentum had been solved. All this, of course, hindered carrying out a detailed analysis and arriving at basic conclusions. But our findings disagree completely with those of Einstein and Klein. Why? Because the work of Einstein and Klein contains a mistake. The two failed to notice that the quantity J_σ with which they operated is simply an identical zero. This very simple mistake is highly important, because it completely destroys Einstein's conclusions.

Let us analyze this question in greater detail. To this end we present Einstein's reasoning and analyze its essence. Einstein, 1918c, wrote:

> ...I wish to demonstrate here that with the help of Eq. (1)* the concepts of energy and momentum can be established as clearly as in classical mechanics. The energy and momentum of a closed system are fully defined independently from the choice of a system of coordinates, provided that the motion of the system (as a whole) with respect to the system of coordinates is fixed; for instance, the "rest energy" of any closed system does not depend on the choice of the system of coordinates.
>
> ...Let us select a system of coordinates such that all linear elements $(0, 0, 0, dx_4)$ are timelike and all linear elements $(dx_1, dx_2, dx_3, 0)$ are spacelike; then the fourth coordinate can in a certain sense be called the "time".
>
> For us to be able to speak of the energy and momentum of a system, the energy and momentum densities must vanish outside a definite region B. This will happen only if outside B the components $g_{\mu\nu}$ are constants, that is, when the system in question is, so to say, immersed in a "Galilean space" and we employ "Galilean coordinates" to describe the surroundings of the system. Region B has infinite dimensions in the direction of the time axis, that is, it intersects any hyperplane $x_4 =$ const. The section of B by a hyperplane $x_4 =$ const is bounded on all sides. Inside B there can be no "Galilean system of coordinates"; the choice of coordinates inside B is limited by a natural condition, namely, that the coordinates must continuously pass into the coordinates outside B. Below I consider some systems of coordinates of this kind, that is, systems coinciding outside B.
>
> The integral laws of conservation of energy and momentum are obtainable from Eq. (1) by integration with respect to x_1, x_2, x_3 over region B. Since at the boundaries of this region all the U_σ^ν vanish, we have
>
> $$\frac{d}{dx_4}\left[\int U_\sigma^4\, dx_1\, dx_2\, dx_3\right] = 0. \tag{3}$$
>
> These four equations express, I believe, the laws of conservation of momentum ($\sigma = 1, 2, 3$) and energy ($\sigma = 4$). Let us denote the integral in Eq. (3) by J_σ. Now I state that the J_σ do not depend on the choice of coordinates for any system of coordinates that outside B coincides with one and the same Galilean system.

Further he noted:

> Thus, notwithstanding the free choice of coordinates inside B, the rest energy or mass of the system constitutes a precisely defined quantity that does not depend on the choice of the system of coordinates. This is even more remarkable because thanks to the nontensor nature of U_σ^ν no invariant interpretation of the components of the energy density can be given.

This reasoning of Einstein contains a simple but fundamental mistake. To verify this, we write the Hilbert-Einstein equation in the form

$$U_\tau^\nu = T_\tau^\nu + t_\tau^\nu = \partial_\mu \sigma_\tau^{\mu\nu}, \tag{3.22}$$

where $\sigma_\tau^{\mu\nu} = -\sigma_\tau^{\nu\mu}$ is the density of an antisymmetric pseudotensor. Substituting Eq. (3.22) into the expression for the 4-momentum of an isolated system, we obtain

$$J_\tau = \int dV\, U_\tau^4 = \int dV\, \partial_\mu\, \sigma_\tau^{\mu 4} = \oint dS_k\, \sigma_\tau^{k4}. \tag{3.23}$$

---

* Formula (1) in Einstein's paper corresponds to Eq. (2.5) in this book.

Since the surface of integration S lies outside B, that is, in a region where all the components of tensor $g_{\mu\nu}$ are constants, the σ_τ^{k4} vanish everywhere on S. This follows directly from the expression (2.14) for σ_τ^{k4}. Thus, (3.23) implies that $J_\tau = 0$. It is this that neither Einstein nor Klein (nor others) noticed. Nor did they understand the correct and profound idea of Hilbert, 1917, (see the Introduction) that in GR there are simply no ordinary conservation laws for energy and momentum. All that followed was completed by the dogmatism and faith that for more than half a century canonized GR, elevating it to an indisputable truth.

Lately Faddeev, 1982, has stated that in GR the Hamiltonian formalism enables solving the problem of the energy-momentum of the gravitational field. But Denisov and Logunov, 1982d, and Denisov and Solov'ev, 1983, have shown that this statement is erroneous and indicates that the author does not understand the essence of the problem.

It is sometimes said that within the framework of GR the gravitational-field energy-momentum tensor can be constructed by replacing the ordinary derivatives in the expression for the pseudotensor with covariant derivatives with respect to "the Minkowski metric". These statements, however, are erroneous. In GR, in contrast to RTG where the Minkowski space-time occupies the center of stage, there can be no global Cartesian coordinates and, hence, we cannot in principle say what form the Minkowski metric γ^{ik} has in GR for a given solution to the Hilbert-Einstein equations. Two solutions of the Hilbert-Einstein equations, say, (3.5) and (3.15), where one is obtained from the other by transforming only the spatial coordinates, have equal status and can, at our choice, be referred to one and the same metric γ^{ik}. But this directly suggests that for each of these solutions we will have different values of the energy of the system. This means that the energy of a system depends on the selection of the spatial coordinates, which is physically meaningless. Such erroneous statements can still be encountered in the literature (e.g. see Ponomarev, 1985).

Chapter 4. Energy-Momentum Conservation in GR

The definition of inertial mass that in Chapter 3 exemplified the groundlessness of a definition based on the energy-momentum pseudotensor does not exhaust all the deficiencies in GR. These deficiencies and the resulting consequences have been discussed in detail in Denisov and Logunov, 1982d. Without going into technical details, we discuss some of them here.

In all physical theories describing the various forms of matter one of the most important field characteristics is the energy-momentum tensor density, which is usually obtained by varying the Lagrangian field density L over the components g_{mn} of the space-time metric tensor.

This characteristic reflects the existence of the field: a physical field exists in a certain region of space-time if and only if the energy-momentum tensor density is nonzero in the region. The energy-momentum of any physical field contributes to the total energy-momentum tensor of the system and does not vanish completely outside the source of field. This enables considering energy transport in the form of waves in the sense of Faraday and Maxwell, that is, we are able to study the distribution of the field's strength in space, determine the energy fluxes through surfaces, calculate the change in energy-momentum in emission and absorption processes, and carry out other energy calculations.

In GR, however, the gravitational field does not possess the properties inherent in other physical fields since it lacks such a characteristic. Indeed, in Einstein's

theory the Lagrangian density consists of two parts: the gravitational-field Lagrangian density $L_g = \sqrt{-g}\, R$, which depends only on the metric tensor g_{mn}, and the material Lagrangian density $L_M = L_M(g_{mn}, \Phi_A)$, which depends on the metric tensor g_{mn} and the other material fields Φ_A. Thus, in Einstein's general theory of relativity the g_{mn} have a two-fold meaning: they are field variables and at the same time components of the metric tensor of space-time.

As a result of such physico-geometrical dualism the density of the total symmetric energy-momentum tensor (the variation of the Lagrangian density over the components of the metric tensor) proves to coincide with the field variables (the variation of the Lagrangian density over the components of the gravitational field). The result is that the density of the total symmetric energy-momentum tensor of the system (field plus matter) is exactly zero:

$$T^{ni} + T^{ni}_{(g)} = 0, \tag{4.0}$$

where $T^{ni} = -2\delta L_M/\delta g_{ni}$ is the density of the symmetric energy-momentum tensor of matter (here by matter we mean all material fields except the gravitational), and

$$T^{ni}_{(g)} = -2\frac{\delta L_g}{\delta g_{ni}} = -\frac{c^4\sqrt{-g}}{8\pi G}\left[R^{ni} - \frac{1}{2}g^{ni}R\right].$$

Equation (4.0) also implies that all the components of the density of the symmetric gravitational-field energy-momentum tensor $T^{ni}_{(g)}$ vanish everywhere outside matter.

Thus, these results alone imply that the gravitational field in Einstein's general theory of relativity does not possess the properties inherent in other physical fields, since outside its source the gravitational field lacks the main physical characteristic, the energy-momentum tensor.

A physical characteristic of a gravitational field in Einstein's theory is the Riemann curvature tensor R^i_{klm}. That we have a clear understanding of this we owe to Synge, 1960 (p. viii):

> If we accept the idea that space-time is a Riemannian four-space (and if we are relativists we must), then surely our first task is to get the feel of it just as early navigators had to get the feel of a spherical ocean. And the first thing we have to get the feel of is the Riemann tensor, for it *is* the gravitational field—if it vanishes, and only then, there is no field. Yet, strangely enough, this most important element has been pushed into the background.

And further he notes (p. ix):

> In Einstein's theory, either there is a gravitational field or there is none, according as the Riemann tensor does not or does vanish. This is an absolute property; it has nothing to do with any observer's world-line.

The absence of such an understanding leads to incomprehension of the essence of Einstein's theory.

Thus, since the gravitational field is characterized solely by the curvature tensor, we cannot introduce in GR a simpler physical characteristic of this field, say, the energy-momentum pseudotensor, with the result that in Einstein's theory energy-momentum pseudotensors are not, in principle, related to the existence of a gravitational field. This assertion has the status of a theorem whose corollary is the possibility of such situations in GR when the curvature tensor is nonzero, that is, a field exists, and yet the energy-momentum pseudotensor vanishes, and vice versa, the curvature tensor vanishes but the energy-momentum pseudotensor is nonzero. Therefore, calculations involving energy-momentum pseudotensors lack all meaning.

Einstein's general theory of relativity links matter and gravitational field into a single entity; while the first is characterized, as in other theories, by an energy-momentum tensor, that is, a second-rank tensor, the second is characterized by the curvature tensor, which is a fourth-rank tensor. Owing to the different dimensions of the physical characteristics of gravitational field and matter in Einstein's theory, it follows directly that there cannot be (in GR) any conservation laws linking matter and gravitational field. This fundamental fact, established in Denisov and Logunov, 1980b, and Hilbert, 1917, means that Einstein's theory was constructed at the expense of repudiating the laws of conservation of matter and gravitational field taken together.

Another physical characteristic of a gravitational field in GR, the Ricci tensor, reflects more the ability of a gravitational field to change the energy-momentum of matter, that is, reflects the action that a gravitational field has on matter, but provides no information on the energy flux carried by a wave. As a result there is no possibility in Einstein's theory of studying the distribution of the strength of a gravitational field in space, of determining the energy fluxes carried by gravitational waves through a surface, etc. That scientists operating within the GR framework can, by employing the idea of pseudotensors, find conserved quantities for matter and gravitational field taken together constitutes a profound delusion.

Indeed, in GR the initial relationship for obtaining conservation laws is the identity

$$\partial_n (T_i^n + \tau_i^n) = 0. \tag{4.1}$$

If matter is concentrated only in a volume V, Eq. (4.1) implies that

$$\frac{d}{dx^0} \int (T_i^0 + \tau_i^0)\, dV = -\oint \tau_i^\alpha\, dS_\alpha. \tag{4.2}$$

At present there exists a whole series of exact solutions to the vacuum Hilbert-Einstein equations for which the stresses τ_0^α are everywhere nil (see Brdička, 1951, Rudakova, 1971, Shirokov, 1970, and Shirokov and Budko, 1967). Consequently, for exact wave solutions to Hilbert-Einstein equations that nullify the components of the energy-momentum pseudotensor, Eq. (4.2) yields

$$\frac{d}{dx^0} \left\{ \int [T_0^0 + \tau_0^0] \right\} dV = 0,$$

that is, the energy of matter and gravitational field inside V is conserved. This means that there is no flow of energy from V outward and, therefore, there can be no action on test bodies placed outside V. This conclusion follows from Einstein's theory.

However, the exact wave solutions to the Hilbert-Einstein equations that nullify the components of the energy-momentum pseudotensor result in a nonzero curvature tensor R^i_{klm}; hence, in view of the equation

$$\frac{\delta^2 n^i}{\delta s^2} + R^i_{klm} u^k u^l n^m = 0, \tag{4.3}$$

where n^i is an infinitesimal geodesic deviation vector, and $u^i = dx^i/ds$ is the 4-vector of velocity, curvature waves act on test bodies lying outside V and change the energy of these bodies. Thus, starting from two different but exact relationships of Einstein's general theory of relativity, we arrive at mutually exclusive physical conclusions.

To understand the reason for these contradictory conclusions let us analyze in more detail the formalism of energy-momentum pseudotensors in Einstein's theory.

Since τ^{ni} is a pseudotensor, by selecting an appropriate system of coordinates we can nullify all the components of τ^{ni} at every point in space. This fact alone raises doubts about interpreting the τ^{ni} as stresses and energy-momentum density of the gravitational field.

It is usually said in this connection (see Møller, 1952) that the gravitational-field energy in GR cannot in principle be localized, that is, that a local distribution of the energy of a gravitational field has no physical meaning since it depends on the choice of the system of coordinates and that only the total energy of closed systems can be well-defined. But such an assertion does not stand up to criticism either.

Indeed, a local distribution of the gravitational-field "energy" defined via any energy-momentum pseudotensor depends on the choice of the system of coordinates and can be nullified at any point in space, which is usually interpreted as the absence of a gravitational-field "energy density" at this point. But a gravitational field described by the curvature tensor cannot be nullified by passing to any admissible system of coordinates. Hence, because curvature waves act on physical processes we cannot state either that in a certain system of coordinates the gravitational field is nil.

This is most clearly seen if we take the example of the exact wave solutions for which the components of the energy-momentum pseudotensor vanish everywhere while the curvature waves do not. And vice versa, in the case of flat space-time, when the metric tensor g_{nl} of the Riemann space-time is equal to the metric tensor γ_{nl} of the pseudo-Euclidean space-time, the components of energy-momentum pseudotensors may not vanish although there is no gravitational field and all the components of the curvature tensor are zero in any system of coordinates.

For example, in the spherical system of coordinates in the pseudo-Euclidean space-time, where $R^i_{klm} = 0$, $g_{00} = 1$, $g_{rr} = -1$, $g_{\theta\theta} = -r^2$, $g_{\varphi\varphi} = -r^2 \sin^2\theta$, we have the following formula for the component τ^0_0 of Einstein's pseudotensor (see Bauer, 1918):

$$\tau^0_0 = -\frac{1}{8\pi}\sin\theta.$$

It is clear that $\tau^0_0 < 0$ and that the total gravitational-field "energy" in this system of coordinates is infinite.

The Landau-Lifshitz pseudotensor in this case demonstrates a different "energy" distribution in space:

$$(-g)\,\tau^{00} = -\frac{r^2}{8\pi}(1 + 4\sin^2\theta).$$

The examples just discussed show that energy-momentum pseudotensors in Einstein's theory do not serve as physical characteristics of the gravitational field and, hence, carry no physical meaning.

Thus, in GR there do not (and cannot) exist any energy-momentum conservation laws for matter and gravitational field taken together. On the other hand, in theories involving other physical fields we have a universal law of energy-momentum conservation and there are no experimental indications of this law being violated. Therefore, we have no reason to reject this law.

So what is the way out of this situation? What can we retain of Einstein's magnificent creation and what must we reject so that in the new theory of gravitation the fundamental law of physics, the law of conservation of energy-momentum of gravitational field and matter taken together, holds true? To answer these questions, we must carefully analyze the deep relationship between energy-momentum conservation laws and the geometry of space-time.

Chapter 5. Energy-Momentum and Angular Momentum Conservation as Related to Geometry of Space-Time

The geometry of space-time largely determines the possibility of obtaining conservation laws for a closed system of interacting fields. As is well known (see Bogoliubov and Shirkov, 1979, and Novozhilov, 1975), a theory for any physical field can be constructed on the basis of the Lagrangian formalism. In this case the physical field is described by a function of coordinates and time, known as the field function, and the equations for determining this function can be found by employing the variational principle of least action. Besides producing field equations, the Lagrangian approach to constructing a classical theory of wave fields makes it possible to derive a number of differential relationships known as differential conservation laws (see Noether, 1918). These relationships follow from the invariance of the action integral under transformations of the space-time coordinates and link the local dynamical characteristics of the field with the respective covariant derivatives in a geometry that is natural in relation to these characteristics.

At present in the literature two types of differential conservation laws are distinguished: strong and weak. Usually a strong conservation law is a differential relationship that holds true solely owing to the invariance of the action integral under coordinate transformations and does not require the existence of equations of motion for the field. A weak conservation law can be obtained from a strong conservation law by allowing for the equations of motion for a system of interacting fields.

It must be stressed that notwithstanding their name, differential conservation laws do not require conservation of some quantity either locally or globally. They are simply differential identities linking the different characteristics of a field and are valid because the action integral does not change under coordinate transformation (i.e. is a scalar). Their name was given by analogy with the differential conservation laws in pseudo-Euclidean space-time, in which differential conservation laws may lead to integral laws. For example, writing the law of conservation for the total energy-momentum tensor of a system of interacting fields in a Cartesian system of coordinates of the pseudo-Euclidean space-time, we get

$$\frac{\partial}{\partial x^0} t^{0i} + \frac{\partial}{\partial x^\alpha} t^{\alpha i} = 0.$$

Integrating over a certain volume and employing the divergence theorem, we get

$$\frac{d}{dx^0} \int t^{0i}\, dV = -\oint t^{\alpha i}\, dS_\alpha.$$

This relationship means that the variation in the energy-momentum of a system of interacting fields inside a certain volume is equal to the energy-momentum flux through the surface enclosing the volume. If this flux is zero, or $\oint t^{\alpha i} dS_\alpha = 0$, we arrive at the conservation law for the total 4-momentum of an isolated system,

$$\frac{d}{dx^0} P^i = 0,$$

where

$$P^i = \frac{1}{c} \int t^{0i}\, dV.$$

Similar integral relationships in the pseudo-Euclidean space-time can be obtained for angular momentum as well.

However, in an arbitrary Riemann space-time the presence of a differential covariant conservation equation does not guarantee the possibility of obtaining a respective integral conservation equation. The possibility of obtaining integral conservation laws in an arbitrary Riemann space-time is totally dependent on the geometry of the space-time and is closely linked with the existence of Killing vectors in the given space-time or, as is sometimes said, with the existence of a group of motions in the Riemann space-time. Let us dwell on this in more detail, since the formalism developed here can be used to obtain integral conservation laws in arbitrary curvilinear systems of coordinates of the pseudo-Euclidean space-time as well.

In an arbitrary Riemann space-time we have the following covariant conservation equation for the total energy-momentum tensor of the system:

$$\nabla_l T^{ml} = \partial_l T^{ml} + \Gamma^m_{nl} T^{nl} + \Gamma^l_{ln} T^{mn} = 0. \tag{5.1}$$

Let us multiply this equation by the Killing vector, that is, a vector η_m that satisfies Killing's equation

$$\nabla_m \eta_n + \nabla_n \eta_m = 0. \tag{5.2}$$

In view of the symmetry of the tensor, $T^{mn} = T^{nm}$, the equation $\nabla_l T^{ml} = 0$ can be written thus:

$$\eta_m \nabla_l T^{ml} = \nabla_l \{\eta_m T^{ml}\} = 0.$$

If we employ the properties of a covariant derivative, we get

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^l} [\sqrt{-g}\, \eta_m T^{ml}] = 0.$$

Since the left-hand side of this equation is a scalar, we can multiply it by $\sqrt{-g}\, dV$ and integrate over a certain volume. We then arrive at the following integral conservation law in the Riemann space-time:

$$\frac{d}{dx^0} \int \sqrt{-g}\, T^{0m} \eta_m \, dV = -\oint \sqrt{-g}\, T^{\alpha m} \eta_m \, dS_\alpha. \tag{5.3}$$

If the flux of the 3-vector through the surface surrounding the volume is nil, then

$$\int \sqrt{-g}\, T^{0m} \eta_m \, dV = \text{const}. \tag{5.3'}$$

Thus, if Killing vectors exist, then from the differential conservation equation (5.1) we can obtain the integral conservation laws (5.3) and (5.3′).

Let us now establish what restrictions must be imposed on the Riemann space-time metric so that Killing's equation (5.2) will have a solution, that is, the conditions that vector η_n must meet so that Eq. (5.2) is satisfied. We note, first, that Killing's equation (5.2) follows from the requirement that the Lie variations of the metric tensor of the Riemann space-time under the infinitesimal coordinate transformations

$$x'^n = x^n + \eta^n(x) \tag{5.4}$$

vanish (here $\eta^n(x)$ is an infinitesimal 4-vector). Indeed, under such a transformation of the coordinates the Lie variation of the metric tensor g_{ln} assumes the form

$$\delta_L g_{ln} = -\nabla_l \eta_n - \nabla_n \eta_l.$$

Comparing this with (5.2), we see that Killing's equation requires that the Lie variation of the metric tensor g_{in} vanish:

$$\delta_L g_{in} = 0.$$

Thus, Killing vectors describe infinitesimal coordinate transformations that leave the metric form-invariant.

Killing's equation (5.2) constitutes a system of first-order partial differential equations. According to the general theory (see Eisenhart, 1933, Petrov, 1966, and Pontryagin, 1966), to establish the solvability conditions for a system of partial differential equations, we must reduce this system to the form

$$\frac{\partial \Theta^a}{\partial x^i} = \Psi_i^a (\Theta^b, x^n), \tag{5.5}$$

where Θ^a are the unknown functions; i, $n = 1, 2, \ldots, N$; and a, $b = 1, 2, \ldots, M$. Then the solvability conditions for system (5.5) can be obtained from the relationship

$$\frac{\partial^2 \Theta^a}{\partial x^i \, \partial x^n} = \frac{\partial^2 \Theta^a}{\partial x^n \, \partial x^i}$$

by replacing the first-order partial derivatives with the right-hand side of Eqs. (5.5):

$$\frac{\partial \Psi_i^a}{\partial x^n} + \frac{\partial \Psi_i^a}{\partial \Theta^b} \Psi_n^b = \frac{\partial \Psi_n^a}{\partial x^i} + \frac{\partial \Psi_n^a}{\partial \Theta^b} \Psi_i^b. \tag{5.6}$$

If the solvability condition (5.6) is met identically in view of the validity of Eqs. (5.5), the system (5.5) is said to be completely integrable and its solution will contain M parameters, the greatest possible number of arbitrary constants for the given system. But if (5.5) is not completely integrable, its solution will contain a smaller number of arbitrary constants. Let us establish the conditions in which the solution to Killing's equations (5.2) in the Riemann space-time V_N contains the greatest possible number of parameters and find this number.

All calculations will be carried out in an explicitly covariant form, which is a generalization of the above scheme for finding the solvability conditions for the system of Killing's equations. We differentiate covariantly Killing's equations (5.2) with respect to parameter x^n. The result is

$$\eta_{i;jn} + \eta_{j;in} = 0.$$

In view of this we have

$$\eta_{i;jn} + \eta_{j;in} + \eta_{i;nj} + \eta_{n;ij} - \eta_{j;ni} - \eta_{n;ji} = 0.$$

Regrouping the terms in this expression, we get

$$\eta_{i;jn} + \eta_{i;nj} + (\eta_{j;in} - \eta_{j;ni}) + (\eta_{n;ij} - \eta_{n;ji}) = 0. \tag{5.7}$$

On the other hand, in view of the commutation relation for covariant derivatives we have

$$\eta_{i;\,nj} - \eta_{i;\,jn} = \eta_k R^k_{inj}. \tag{5.8}$$

If we substitute this into (5.7), we get

$$2\eta_{i;\,jn} + \eta_k R^k_{inj} + \eta_k R^k_{jin} + \eta_k R^k_{nij} = 0. \tag{5.9}$$

Using Ricci's identity

$$R^k_{inl} + R^k_{nli} + R^k_{lin} = 0, \tag{5.10}$$

we get

$$\eta_k R^k_{inj} + \eta_k R^k_{jin} = \eta_k R^k_{nij},$$

which means that we can write (5.9) in the following form:

$$\eta_{i;\,jn} = -\eta_k R^k_{nij}.$$

We have, therefore, arrived at the following covariant equations:

$$\eta_{l;\,n} + \eta_{n;\,i} = 0, \quad \eta_{i;\,jn} = -\eta_k R^k_{nij}. \tag{5.11}$$

Let us transform this system of covariant differential equations into a system containing only first covariant derivatives. To this end, in addition to the N unknown components of vector η_m we introduce an unknown tensor λ_{lm} that obeys the equation

$$\eta_{l;m} = \lambda_{lm}. \tag{5.12}$$

This tensor contains N^2 unknown components, but only $N(N-1)/2$ of these are independent since this tensor is antisymmetric in view of Eqs. (5.2) and (5.12):

$$\lambda_{ml} + \lambda_{lm} = 0. \tag{5.13}$$

If we allow for all this, the sought system of covariant differential equations assumes the form

$$\eta_{m;\,l} = \lambda_{ml}, \quad \lambda_{ml;\,j} = \eta_k R^k_{jlm}. \tag{5.14}$$

We have, therefore, reduced Killing's equations (5.2) to a system of a special type consisting of linear differential equations in first-order covariant derivatives.

This system is a covariant generalization of system (5.5), with the unknown functions Θ^α being the $N(N+1)/2$ components of tensors η_m and λ_{ml}:

$$\Theta^\alpha = \{\eta_m, \lambda_{ml}\}.$$

The solvability condition for system (5.14) can be obtained from the commutation relation for covariant derivatives, which follows from the independence of the order in which derivatives are taken in partial differentiation. On the basis of this rule we get

$$\begin{gathered} \eta_{i;\,mj} - \eta_{i;\,jm} = \eta_k R^k_{imj}, \\ \lambda_{lm;\,jl} - \lambda_{lm;\,lj} = \lambda_{lk} R^k_{mjl} + \lambda_{km} R^k_{ijl}. \end{gathered} \tag{5.15}$$

Replacing the first covariant derivatives on the left-hand sides of (5.15) with their expressions (5.14) and employing (5.13), which reflects the fact that λ_{lm} is antisymmetric, we arrive at the solvability conditions for system (5.14) in the form

$$\lambda_{lm;\,j} - \lambda_{lj;\,m} = \eta_k R^k_{imj}, \tag{5.16}$$

$$[\eta_k R^k_{jmi}]_{;\,l} - [\eta_k R^k_{lmi}]_{;\,j} = \lambda_{ik} R^k_{mjl} + \lambda_{km} R^k_{ijl}. \tag{5.17}$$

It is easily verified that (5.16) is satisfied identically because of the validity of Eqs. (5.14) and the properties of the curvature tensor. Thus, if condition (5.17) is satisfied identically solely because of the symmetry properties of the Riemann space-time, then system (5.14) will be completely integrable and, hence, the solution to Killing's equations (5.2) will contain the greatest possible number $M =$

$N(N+1)/2$ of arbitrary constants. Since the unknown functions η_m and $\lambda_{ml} = -\lambda_{lm}$ entering into system (5.14) must be independent in this case, the left-hand side of (5.17) vanishes identically only if

$$R^h_{mij;\,l} - R^h_{lij;\,m} = 0, \tag{5.18}$$

$$\delta^n_j R^h_{iml} - \delta^h_j R^n_{iml} - \delta^n_i R^h_{jml} + \delta^h_i R^n_{jml} + \delta^n_l R^h_{mij} - \delta^h_l R^n_{mij} - \delta^n_m R^h_{lij} + \delta^h_m R^n_{lij} = 0. \tag{5.19}$$

If we contract (5.19) on l and n and allow for the relationships $R^n_{lmn} = R_{lm}$ and $R^n_{nml} = 0$ and for Ricci's identity (5.10), we get

$$(N-1)\, R^h_{mij} = \delta^h_j R_{mi} - \delta^h_i R_{jm}.$$

From this it follows that

$$R_{lmij} = \frac{1}{N-1}\,(g_{jl}R_{mi} - g_{il}R_{jm}). \tag{5.20}$$

Multiplying this equation into g^{mi}, we get

$$NR_{jl} = g_{jl}R.$$

If we now substitute this into (5.20), we arrive at a condition in view of which (5.19) is satisfied identically:

$$R_{lmij} = \frac{R}{N(N-1)}\,[g_{jl}g_{mi} - g_{il}g_{jm}]. \tag{5.21}$$

Combining (5.21) and Eq. (5.18) results in a requirement that the scalar curvature must satisfy:

$$[\delta^k_j g_{im} - \delta^k_i g_{jm}]\,\frac{\partial}{\partial x^l} R - [\delta^k_j g_{il} - \delta^k_i g_{lj}]\,\frac{\partial}{\partial x^m} R = 0.$$

If we multiply this by $\delta^l_k g^{mi}$, we get

$$(N-1)\,\frac{\partial R}{\partial x^j} = 0.$$

Since in the case considered $N > 1$, the above condition is met if and only if $R = \text{const}$. Hence, the solvability conditions (5.18) and (5.19) for Killing's equations (5.2) will be satisfied identically if and only if the Riemann space-time curvature tensor has the form

$$R_{lmij} = \frac{R}{N(N-1)}\,[g_{jl}g_{mi} - g_{il}g_{jm}],$$

with $R = \text{const}$.

Thus, Killing's equations have solutions containing the greatest possible number $M = N(N+1)/2$ of arbitrary constants (parameters) if and only if the Riemann space-time V_N is a constant-curvature space, and if V_N is not a constant-curvature space, the number of parameters will be smaller than M.

Hence, mathematically speaking, the presence of integral conservation laws for energy-momentum and angular momentum reflects the existence of certain properties inherent in space-time—its homogeneity and isotropy. There are three types of four-dimensional spaces that possess the properties of homogeneity and isotropy to an extent that allows for introducing ten integrals of motion for a closed system. These are the space with constant negative curvature (Lobachevskii's space, or hyperbolic space), the zero-curvature space (pseudo-Euclidean space), and the space with constant positive curvature (Riemann space). The

first two spaces are infinite, with an infinite volume, while the third is closed, with a finite volume, but has no boundaries.

Let us now find the Killing vector in an arbitrary curvilinear system of coordinates of the pseudo-Euclidean space-time. To this end we first write Killing's equations in the Cartesian system of coordinates:

$$\partial_i \eta_n + \partial_n \eta_i = 0.$$

Hence, to determine a Killing vector we have a system of ten first-order partial linear differential equations. Solving this system according to general rules, we obtain

$$\eta_l = a_l + \omega_{lm} x^m, \tag{5.22}$$

where a_l is an arbitrary constant infinitesimal vector, and ω_{lm} is an arbitrary constant infinitesimal tensor satisfying the condition

$$\omega_{lm} = -\omega_{ml}.$$

Thus, solution (5.22), as expected, contains all ten arbitrary parameters. Since (5.22) contains ten independent parameters, we actually have ten independent Killing vectors, and (5.22) constitutes a linear combination of the ten independent vectors.

Let us establish the meaning of these parameters. Substituting (5.22) into (5.4), we get

$$x'^n = x^n + a^n + \omega^n_m x^m. \tag{5.23}$$

We see that the four parameters a^n are components of the 4-vector of infinitesimal translations of the reference frame. The three parameters $\omega_{\alpha\beta}$ are the components of the tensor of rotation through an infinitesimal angle about a certain axis (the so-called proper rotation). The three parameters $\omega_{0\beta}$ describe infinitesimal rotations in the (x^0, x^β)-plane, known as Lorentzian rotations. Since the metric tensor γ_{mn} is form-invariant under translations, the pseudo-Euclidean space-time is homogeneous; its properties do not depend on the position of the origin in the space. Similarly, the form-invariance of the metric tensor γ_{mn} under four-dimensional rotations leads to the isotropy of this space, which means that all directions in the pseudo-Euclidean space-time have equal status.

Thus, the pseudo-Euclidean space-time admits of a ten-parameter group of motions consisting of a four-parameter translation subgroup and a six-parameter rotation subgroup. The existence of this group of motions and the corresponding Killing vectors guarantees ten integral laws of conservation of energy-momentum and angular momentum in a system of interacting fields. Indeed, allowing for the fact that in the Cartesian system of coordinates $\sqrt{-\gamma} = 1$ and for the general relationship (5.3), we find that in the case of the translation subgroup $(\eta_i = a_i)$

$$\frac{d}{dx^0} \int T^{0m} a_m \, dV = -\oint dS_\alpha T^{\alpha m} a_m.$$

Since a_m is an arbitrary constant vector, this relationship yields

$$\frac{d}{dx^0} \int T^{0m} \, dV = -\oint dS_\alpha T^{\alpha m}.$$

For an isolated system of interacting fields, the expression on the right-hand side of this relationship vanishes, as a result of which the total 4-momentum of the

system is conserved:

$$P^m = \int T^{0m}\, dV = \text{const.} \tag{5.24}$$

Similarly, at $\eta_n = \omega_{nm} x^m$ we get

$$\frac{d}{dx^0} \int dV\, T^{0m} x^n \omega_{mn} = -\oint dS_\alpha T^{\alpha m} x^n \omega_{mn}.$$

Since the constant tensor ω_{mn} is antisymmetric, the above leads us to the following integral conservation law for angular momentum

$$\frac{d}{dx^0} \int dV\, [T^{0m} x^n - T^{0n} x^m] = -\oint dS_\alpha\, [T^{\alpha m} x^n - T^{\alpha n} x^m]. \tag{5.25}$$

For an isolated system the total angular momentum is conserved because the right-hand side of (5.25) vanishes:

$$M^{mn} = \int dV\, [T^{0m} x^n - T^{0n} x^m] = \text{const.} \tag{5.26}$$

Only in the pseudo-Euclidean space-time are there separate laws of conservation for energy-momentum and angular momentum of a closed system.

Note that we can obtain the solution to Killing's equations (5.2) in arbitrary curvilinear coordinates of the pseudo-Euclidean space-time in view of the tensor nature of x^i and η^i from the solution (5.23) to these equations in the Cartesian coordinate system. To this end we transfer in (5.23) from Cartesian coordinates x^i to arbitrary curvilinear coordinates x_{N}^i thus:

$$x^i = f^i(x_{\mathrm{N}}).$$

This yields

$$\eta_m^{\mathrm{N}} = \frac{\partial f^i}{\partial x_{\mathrm{N}}^m} \eta_i [x(x_{\mathrm{N}})].$$

Thus, in an arbitrary curvilinear coordinate system of the pseudo-Euclidean space-time, the Killing vectors have the form

$$\eta_m^{\mathrm{N}} = \frac{\partial f^i(x_{\mathrm{N}})}{\partial x_{\mathrm{N}}^m} a_i + \frac{\partial f^i(x_{\mathrm{N}})}{\partial x_{\mathrm{N}}^m} \omega_{in} f^n(x_{\mathrm{N}}). \tag{5.27}$$

It is not very difficult to generalize Eqs. (5.24)-(5.26) so that they incorporate the case of arbitrary curvilinear coordinates. Proceeding in the same manner as we did above, we arrive at the following expression for the 4-momentum of an isolated system:

$$P^i = \int \sqrt{-\gamma(x_{\mathrm{N}})}\, dx_{\mathrm{N}}^1\, dx_{\mathrm{N}}^2\, dx_{\mathrm{N}}^3\, \frac{\partial f^i(x_{\mathrm{N}})}{\partial x_{\mathrm{N}}^m} T^{0m}(x_{\mathrm{N}}).$$

The antisymmetric tensor of angular momentum in this case has the form

$$M^{im} = \int \sqrt{-\gamma(x_{\mathrm{N}})}\, dx_{\mathrm{N}}^1\, dx_{\mathrm{N}}^2\, dx_{\mathrm{N}}^3\, T^{0n}(x_{\mathrm{N}}) \left[f^m(x_{\mathrm{N}}) \frac{\partial f^i(x_{\mathrm{N}})}{\partial x_{\mathrm{N}}^n} - f^i(x_{\mathrm{N}}) \frac{\partial f^m(x_{\mathrm{N}})}{\partial x_{\mathrm{N}}^n} \right].$$

Thus, the geometry of space-time determines the possibility of obtaining integral conservation laws. In the case of four dimensions (the physical space-time) only spaces with constant curvature possess all ten integral conservation laws; in other spaces the number of these laws is less.

Our analysis demonstrates that if we wish to have the greatest number of conserved quantities, we must reject Riemannian geometry in its general form, and for all fields, including the gravitational, we must select one of the above-mentioned geometries of constant curvature as the natural one. Since the existing experimental data on strong, weak, and electromagnetic interactions suggests that for the fields related to these interactions the natural geometry of space-time is pseudo-Euclidean, we can assume at least at the present level of our knowledge that this geometry is the universal natural geometry for all physical processes, including those involving gravitation.

This assertion constitutes one of the main theses of our approach to the theory of gravitational interaction. It obviously leads to the observance of all laws of conservation of energy-momentum and angular momentum and ensures the existence of all ten integrals of motion for a system consisting of a gravitational field and other material fields.

As we will shortly show, the gravitational field in our framework, as all other physical fields, is characterized by an energy-momentum tensor that contributes to the total tensor of energy-momentum of the system. This constitutes the main difference between our approach and Einstein's. It must also be noted that in the pseudo-Euclidean space-time the integration of tensor quantities, in addition to its general simplicity, has a well-defined meaning.

Another key issue that emerges in constructing a theory of the gravitational field is the question of the way in which the field interacts with matter. In acting on matter, a gravitational field changes the geometry of matter if it enters into terms in the highest-order derivatives in the equations of motion of the matter. Then the motion of material bodies and other physical fields in the pseudo-Euclidean space-time under the action of the gravitational field can in no way be distinguished from their motion in an effective Riemann space-time.

Experimental data suggests that the action of a gravitational field on matter is universal. This led us to formulate the geometrization principle (see the Introduction). Hence, the effective Riemann space-time will be universal for all forms of matter (see Appendix 5).

The geometrization principle was formulated in Denisov and Logunov, 1982a, 1982c, Denisov, Logunov, and Mestvirishvili, 1981a, and Logunov, Denisov, Vlasov, Mestvirishvili, and Folomeshkin, 1979, but actually the idea was first put forward in Logunov and Folomeshkin, 1977. The principle means that the description of the motion of matter under the action of a gravitational field in a pseudo-Euclidean space-time is physically identical to the description of the motion of matter in the appropriate effective Riemann space-time. In this approach the gravitational field (as a physical field) is excluded, so to say, from the description of the motion of matter, and the field's energy, figuratively speaking, is spent on forming the effective Riemann space-time.

Thus, the effective Riemann space-time is a peculiar carrier of energy-momentum. The amount of energy used for creating this space-time is exactly equal to the amount contained in the gravitational field; hence, the propagation of curvature waves in the Riemann space-time reflects common energy transfer via gravitational waves in the pseudo-Euclidean space-time. This means that in our approach the existence of curvature waves in the Riemann space-time follows directly from the existence of gravitational waves in the sense of Faraday and Maxwell, waves that carry an energy-momentum density.

We note also that when we introduce the geometrization principle, we thereby retain Einstein's idea of the Riemannian geometry of space-time for matter. This does not mean, however, that we must inevitably return to GR. The general theory

of relativity constitutes a partial realization of this idea, rather than the other way round. Hence, the idea of a gravitational field as a physical field that can carry energy changes our conceptions about space-time and gravity. The relativistic theory of gravitation, which realizes this idea, makes it possible to describe the entire body of data on gravitational experiments, satisfies the correspondence principle, and leads to a number of fundamental corollaries.

Chapter 6. The Geometrization Principle and General RTG Relations

Without loss of generality, let us assume that the tensor density $\tilde{g}^{ik}$ of the metric tensor of the Riemann space-time is a local function that depends on the density $\tilde{\gamma}^{ik}$ of the metric tensor of the Minkowski space-time and the density $\tilde{\Phi}^{ik}$ of the gravitational-field tensor. We assume that the material Lagrangian density L_M is dependent only on the fields Φ_A, on their first-order covariant derivatives, and, in view of the geometrization principle, on $\tilde{g}^{ik}$. We also assume that the gravitational-field Lagrangian density depends on $\tilde{\gamma}^{ik}$, on the first-order partial derivatives of $\tilde{\gamma}^{ik}$, on $\tilde{\Phi}^{ik}$, and on the first-order covariant derivatives of $\tilde{\Phi}^{ik}$ with respect to the Minkowski metric. To derive conservation laws we employ the invariance of the action integral under infinitesimal translations of the coordinates. Since for every given Lagrangian density L the action integral $J = \int L\, d^4x$ is a scalar, under an arbitrary infinitesimal coordinate transformation the variation δJ vanishes. Let us start by calculating the variation of the material action integral $J_M = \int L_M\, d^4x$ brought on by the transformation

$$x'^i = x^i + \xi^i(x), \tag{6.1}$$

where $\xi^i(x)$ is an infinitesimal 4-vector of displacement:

$$\delta J_M = \int d^4x \left[\frac{\delta L_M}{\delta \tilde{g}^{mn}} \delta_L \tilde{g}^{mn} + \frac{\delta L_M}{\delta \Phi_A} \delta_L \Phi_A + \mathrm{div} \right] = 0. \tag{6.2}$$

Here div stands for the divergence terms, which in the present chapter play no role in our discussion.

The Eulerian variation is defined in the usual way:

$$\frac{\delta L}{\delta \varphi} \equiv \frac{\partial L}{\partial \varphi} - \partial_n \frac{\partial L}{\partial(\partial_n \varphi)} + \partial_n \partial_k \frac{\partial L}{\partial(\partial_n \partial_k \varphi)} - \ldots .$$

The variations $\delta_L \tilde{g}^{mn}$ and $\delta_L \Phi_A$ generated by the coordinate transformation (6.1) can easily be calculated if we employ the transformation laws:

$$\delta_L \tilde{g}^{mn} = \tilde{g}^{kn} D_k \xi^m + \tilde{g}^{km} D_k \xi^n - D_k (\xi^k \tilde{g}^{mn}), \tag{6.3}$$

$$\delta_L \Phi_A = -\xi^k D_k \Phi_A + F_{A;\,k}^{B;\,n} \Phi_B D_n \xi^k. \tag{6.4}$$

Here and in what follows the D_k are the covariant derivatives with respect to the Minkowski metric. Substituting (6.3) and (6.4) into (6.2) and integrating by parts,

we get

$$\delta J_M = \int d^4x \left\{ -\xi^m \left[D_k \left(2 \frac{\delta L_M}{\delta \tilde{g}^{mn}} \tilde{g}^{kn} \right) - D_m \left(\frac{\delta L_M}{\delta \tilde{g}^{lp}} \right) \tilde{g}^{lp} + D_k \left(\frac{\delta L_M}{\delta \Phi_A} F^{B;\,k}_{A;\,m} \Phi_B \right) + \frac{\delta L_M}{\delta \Phi_A} D_m \Phi_A \right] + \mathrm{div} \right\} = 0.$$

Since the vector ξ^m is arbitrary, the condition $\delta J_M = 0$ yields the following strong identity:

$$D_k \left(2 \frac{\delta L_M}{\delta \tilde{g}^{mn}} \tilde{g}^{kn} \right) - D_m \left(\frac{\delta L_M}{\delta \tilde{g}^{lp}} \right) \tilde{g}^{lp} = -D_k \left(\frac{\delta L_M}{\delta \Phi_A} F^{B;\,k}_{A;\,m} \Phi_B \right) - \frac{\delta L_M}{\delta \Phi_A} D_m \Phi_A, \quad (6.5)$$

which is valid irrespective of whether or not the equations of motion of the fields are valid.

Let us introduce the following notation:

$$T_{mn} = 2 \frac{\delta L_M}{\delta g^{mn}}, \quad T^{mn} = -2 \frac{\delta L_M}{\delta g_{mn}} = g^{mk} g^{np} T_{kp}, \quad (6.6a)$$

$$\tilde{T}_{mn} = 2 \frac{\delta L_M}{\delta \tilde{g}^{mn}}, \quad \tilde{T}^{mn} = -2 \frac{\delta L_M}{\delta \tilde{g}_{mn}} = \tilde{g}^{mk} \tilde{g}^{np} \tilde{T}_{kp}, \quad (6.6b)$$

where T_{mn} is the material energy-momentum tensor density in the Riemann space-time and is known as the Hilbert-tensor density.

If we allow for (6.6b), we can represent the left-hand side of (6.5) in the following form:

$$D_k (\tilde{T}_{mn} \tilde{g}^{kn}) - \frac{1}{2} \tilde{g}^{kp} D_m \tilde{T}_{kp} = \partial_k (\tilde{T}_{mn} \tilde{g}^{kn}) - \frac{1}{2} \tilde{g}^{kp} \partial_m \tilde{T}_{kp}.$$

The right-hand side of this equation can easily be reduced to

$$\partial_k (\tilde{T}_{mn} \tilde{g}^{kn}) - \frac{1}{2} \tilde{g}^{kp} \partial_m \tilde{T}_{kp} = \tilde{g}_{mn} \nabla_k \left(\tilde{T}^{kn} - \frac{1}{2} \tilde{g}^{kn} \tilde{T} \right), \quad (6.7)$$

where $\tilde{T} = \tilde{g}_{kp} \tilde{T}^{kp}$, and ∇_k is the symbol of covariant differentiation with respect to the metric of the Riemann space-time.

On the basis of (6.7) we can now write the strong identity (6.5) in the following form:

$$\tilde{g}_{mn} \nabla_k \left(\tilde{T}^{kn} - \frac{1}{2} \tilde{g}^{kn} \tilde{T} \right) = -D_k \left(\frac{\delta L_M}{\delta \Phi_A} F^{B;\,k}_{A;\,m} \Phi_B \right) - \frac{\delta L_M}{\delta \Phi_A} D_m \Phi_A. \quad (6.8)$$

In view of the principle of least action the equations of motion for the material fields have the form

$$\frac{\delta L_M}{\delta \Phi_A} = 0. \quad (6.9)$$

Combining this with (6.8) results in the weak identity

$$\nabla_m \left(\tilde{T}^{mn} - \frac{1}{2} \tilde{g}^{mn} \tilde{T} \right) = 0. \quad (6.10)$$

Note that the energy-momentum tensor density for matter, T^{mn}, in the Riemann space-time is related to $\tilde{T}^{mn}$ in the following manner:

$$\sqrt{-g}\, T^{mn} = \tilde{T}^{mn} - \frac{1}{2} \tilde{g}^{mn} \tilde{T}. \quad (6.11)$$

Hence, (6.10) results in the following covariant equation of conservation of matter in the Riemann space-time:

$$\nabla_m T^{mn} = 0. \tag{6.12}$$

Only if the number of equations for a material field is four can we use instead of the equations (6.9) for this field the equivalent equations (6.12). The variation of the action integral (6.2) can be written in the equivalent form

$$\delta J_{\mathrm{M}} = \int d^4x \left\{ \frac{\delta L_{\mathrm{M}}}{\delta \widetilde{\Phi}^{mn}} \delta_{\mathrm{L}} \widetilde{\Phi}^{mn} + \frac{\delta L_{\mathrm{M}}}{\delta \widetilde{\gamma}^{mn}} \delta_{\mathrm{L}} \widetilde{\gamma}^{mn} + \frac{\delta L_{\mathrm{M}}}{\delta \Phi_A} \delta_{\mathrm{L}} \Phi_A + \mathrm{div} \right\} = 0, \tag{6.13}$$

where the variations $\delta_{\mathrm{L}}\widetilde{\Phi}^{mn}$ and $\delta_{\mathrm{L}}\widetilde{\gamma}^{mn}$ generated by the coordinate transformation (6.1) are

$$\delta_{\mathrm{L}}\widetilde{\Phi}^{mn} = \widetilde{\Phi}^{kn} D_k \xi^m + \widetilde{\Phi}^{km} D_k \xi^n - D_k (\xi^k \widetilde{\Phi}^{mn}), \tag{6.14}$$

$$\delta_{\mathrm{L}}\widetilde{\gamma}^{mn} = \widetilde{\gamma}^{kn} D_k \xi^m + \widetilde{\gamma}^{km} D_k \xi^n - \widetilde{\gamma}^{mn} D_k \xi^k. \tag{6.15}$$

Substituting the expressions for the variations $\delta_{\mathrm{L}}\widetilde{\Phi}^{mn}$, $\delta_{\mathrm{L}}\widetilde{\gamma}^{mn}$, and $\delta_{\mathrm{L}}\Phi_A$ into (6.13) and integrating by parts, we arrive, in view of the arbitrariness of ξ^m, at the following strong identity:

$$\begin{aligned} D_k \left(2 \frac{\delta L_{\mathrm{M}}}{\delta \widetilde{\Phi}^{mn}} \widetilde{\Phi}^{kn} \right) - D_m \left(\frac{\delta L_{\mathrm{M}}}{\delta \widetilde{\Phi}^{kp}} \right) \widetilde{\Phi}^{kp} + D_k \left(2 \frac{\delta L_{\mathrm{M}}}{\delta \widetilde{\gamma}^{mn}} \right) \widetilde{\gamma}^{kn} \\ - D_m \left(\frac{\delta L_{\mathrm{M}}}{\delta \widetilde{\gamma}^{kp}} \widetilde{\gamma}^{kp} \right) = - D_k \left(\frac{\delta L_{\mathrm{M}}}{\delta \Phi_A} F^{B;\,k}_{A;\,m} \Phi_B \right) - \frac{\delta L_{\mathrm{M}}}{\delta \Phi_A} D_m \Phi_A, \end{aligned} \tag{6.16}$$

which, like (6.5), is valid irrespective of whether or not the equations of the motion of matter and gravitational field are valid.

For an arbitrary Lagrangian we introduce several notations and relationships that will be used later:

$$\widetilde{t}^{mn} = -2 \frac{\delta L}{\delta \widetilde{\gamma}_{mn}}, \quad t^{mn} = -2 \frac{\delta L}{\delta \gamma_{mn}}, \tag{6.17a}$$

$$t^{mn} = \frac{1}{\sqrt{-\gamma}} \left(\widetilde{t}^{mn} - \frac{1}{2} \widetilde{\gamma}^{mn} \widetilde{t} \right). \tag{6.17b}$$

Since L_{M} depends, in view of the geometrization principle, on $\widetilde{\gamma}^{mn}$ only through $\widetilde{g}^{mn}$, we can easily find the relationship linking $\widetilde{T}_{mn}$ and $\widetilde{t}_{(\mathrm{M})mn}$:

$$\widetilde{t}_{(\mathrm{M})mn} = 2 \frac{\delta L_{\mathrm{M}}}{\delta \widetilde{\gamma}^{mn}} = \widetilde{T}_{kp} \frac{\partial \widetilde{g}^{kp}}{\partial \widetilde{\gamma}^{mn}}, \tag{6.18a}$$

where we have allowed for definition (6.6b). Taking into account the identity

$$\frac{\partial \widetilde{g}^{kp}}{\partial \widetilde{\gamma}_{mn}} = - \widetilde{\gamma}^{ml} \widetilde{\gamma}^{nq} \frac{\partial \widetilde{g}^{pk}}{\partial \widetilde{\gamma}^{lq}}$$

and combining it with (6.17a), we get

$$\widetilde{t}^{mn}_{(\mathrm{M})} = - \widetilde{T}_{pk} \frac{\partial \widetilde{g}^{pk}}{\partial \widetilde{\gamma}_{mn}}. \tag{6.18b}$$

Allowing in (6.18b) for identity (6.6b) and for the fact that

$$-\tilde{g}_{lp}\tilde{g}_{qk}\frac{\partial \tilde{g}^{lq}}{\partial \tilde{\gamma}_{mn}}=\frac{\partial \tilde{g}_{pk}}{\partial \tilde{\gamma}_{mn}},$$

we find that (6.18b) yields

$$\tilde{t}^{mn}_{(\mathrm{M})}=\tilde{T}^{pk}\frac{\partial \tilde{g}_{pk}}{\partial \tilde{\gamma}_{mn}}. \tag{6.18c}$$

Now, if we compare the identities (6.8) and (6.16) and allow for (6.17a), we obtain

$$\tilde{g}_{mn}\nabla_k\left(\tilde{T}^{kn}-\frac{1}{2}\tilde{g}^{kn}\tilde{T}\right)=\tilde{\gamma}_{mn}D_k\left(\tilde{t}^{kn}_{(\mathrm{M})}-\frac{1}{2}\tilde{\gamma}^{kn}\tilde{t}_{(\mathrm{M})}\right)+D_k\left(2\frac{\delta L_{\mathrm{M}}}{\delta\tilde{\Phi}^{mn}}\tilde{\Phi}^{kn}\right)$$
$$-D_m\left(\frac{\delta L_{\mathrm{M}}}{\delta\tilde{\Phi}^{kp}}\right)\tilde{\Phi}^{kp}. \tag{6.19}$$

Similarly, from the invariance of the gravitational-field action integral $J_g=\int L_g\,d^4x$ under coordinate transformations (6.1) it follows that

$$\tilde{\gamma}_{mn}D_k\left(\tilde{t}^{kn}_{(g)}-\frac{1}{2}\tilde{\gamma}^{kn}\tilde{t}_{(g)}\right)+D_k\left(2\frac{\delta L_g}{\delta\tilde{\Phi}^{mn}}\tilde{\Phi}^{kn}\right)-D_m\left(\frac{\delta L_g}{\delta\tilde{\Phi}^{kp}}\right)\tilde{\Phi}^{kp}=0. \tag{6.20}$$

Adding (6.19) to (6.20), we get

$$\tilde{g}_{mn}\nabla_k\left(\tilde{T}^{kn}-\frac{1}{2}\tilde{g}^{kn}\tilde{T}\right)=\tilde{\gamma}_{mn}D_k\left(\tilde{t}^{kn}-\frac{1}{2}\tilde{\gamma}^{kn}\tilde{t}\right)+D_k\left(2\frac{\delta L}{\delta\tilde{\Phi}^{mn}}\tilde{\Phi}^{kn}\right)$$
$$-D_m\left(\frac{\delta L}{\delta\tilde{\Phi}^{kp}}\right)\tilde{\Phi}^{kp}. \tag{6.21}$$

Here and in what follows

$$\tilde{t}^{kn}=\tilde{t}^{kn}_{(g)}+\tilde{t}^{kn}_{(\mathrm{M})}. \tag{6.22}$$

Owing to the principle of least action, the equations for the gravitational field assume the form

$$\frac{\delta L}{\delta\tilde{\Phi}^{mn}}=\frac{\delta L_g}{\delta\tilde{\Phi}^{mn}}+\frac{\delta L_{\mathrm{M}}}{\delta\tilde{\Phi}^{mn}}=0. \tag{6.23}$$

Allowing for these equations, we see that (6.21) yields the most important equality:

$$\tilde{g}_{mn}\nabla_k\left(\tilde{T}^{kn}-\frac{1}{2}\tilde{g}^{kn}\tilde{T}\right)=\tilde{\gamma}_{mn}D_k\left(\tilde{t}^{kn}-\frac{1}{2}\tilde{\gamma}^{kn}\tilde{t}\right). \tag{6.24}$$

Since the density of the total energy-momentum tensor in the Minkowski space-time is given by the formula

$$\sqrt{-\gamma}\,t^{kn}=\tilde{t}^{kn}-\frac{1}{2}\tilde{\gamma}^{kn}t, \tag{6.25}$$

combining this expression with (6.11) we find that (6.24) can be written in the following form:

$$D_m t^m_n=\nabla_m T^m_n. \tag{6.26}$$

This formula represents the geometrization principle, namely, that the covariant divergence in the pseudo-Euclidean space of the sum of the tensor densities of

energy-momentum of matter and gravitational field taken together is exactly equal to the covariant divergence in the effective Riemann space-time of only the energy-momentum tensor density of matter. If the equations of motion of matter hold true, we have

$$D_m t_n^m = \nabla_m T_n^m = 0. \tag{6.27}$$

In our discussion we have assumed that the equations of motion of matter are not corollaries of the equations (6.23) for the gravitational field, since only in this case will the system of equations (6.23), (6.27) be complete for determining the material and gravitational-field variables. The covariant equation of matter conservation in the Riemann space-time does not provide a clear picture of which quantity is conserved, while the law of conservation of the total energy-momentum tensor t_n^m in the Minkowski space-time clearly states that the energy-momentum of matter and gravitational field taken together is conserved. Thus, in the present theory the Riemann space-time emerges as a result of the action of the gravitational field on all forms of matter, hence this space-time is the effective Riemann space-time of field origin. The Minkowski space-time finds its precise physical reflection in the laws of conservation of the tensors of energy-momentum and angular momentum of matter and gravitational field taken together.

Since in flat space-time there are ten Killing vectors, there must be ten conserved integral quantities for a closed system of fields. Also, since the equation that reflects the conservation of the total energy-momentum tensor in the Minkowski space-time,

$$D_m t_n^m = D_m (t_{n(g)}^m + t_{n(M)}^m) = 0, \tag{6.28}$$

is equivalent to the covariant equation representing matter conservation in the Riemann space-time, and the latter is equivalent to the equations of motion of matter, we can use Eq. (6.28) instead of the equation of motion of matter.

It must be especially noted that both matter and gravitational field are characterized in the given theory by energy-momentum tensors and, therefore, in contrast to GR, there cannot in principle emerge any pseudotensors, with the result that all nonphysical conceptions about the impossibility of localizing the gravitational field are absent from our theory.

If we were to take, following Hilbert and Einstein, the gravitational-field Lagrangian density in a completely geometrized form, that is, depending only on the metric tensor g^{ik} of the Riemann space-time and its derivatives, say $L_g = \sqrt{-g}\, R$, with R the scalar curvature of the Riemann space-time, then the energy-momentum tensor density of a free gravitational field in the Minkowski space-time would, in view of the field equations, vanish everywhere:

$$\frac{\delta L_g}{\delta \gamma^{mn}} = \frac{\delta L_g}{\delta g^{pk}} \frac{\partial g^{pk}}{\partial \gamma^{mn}} = 0. \tag{6.29}$$

Thus, if we take the Minkowski space-time and a physical tensor field possessing energy and momentum, we cannot in principle build a completely geometrized gravitational-field Lagrangian. Therefore, a theory based on a completely geometrized Lagrangian cannot in principle describe a physical gravitational field in the sense of Faraday and Maxwell in the Minkowski space-time. It has been stated in the literature (e.g. see Ogievetsky and Polubarinov, 1965a, 1965b) that employing a tensor field with spin 2 in the Minkowski space-time results unambiguously in a GR gravitational-field Lagrangian equal to R. However, such

statements carry no physical meaning because the gravitational-field energy-momentum tensor introduced in the argument is zero, as (6.29) clearly shows. Therefore, such research is physically meaningless and the results are erroneous.

Chapter 7. The Basic Identity

As shown in Barnes, 1965, and Fronsdal, 1958, the symmetric second-rank tensor f^{ik} can be expanded in a direct sum of irreducible representations, one with spin 2, one with spin 1, and two with spin 0:

$$f^{lm}=[P_2+P_1+P_0+P_{0'}]^{lm}_{ik}f^{ik}, \tag{7.1}$$

where by P_s, $s=2, 1, 0, 0'$, we denote the projection operators, which satisfy the following standard relationships:

$$P_sP^t=\delta^t_sP_t \text{ (here there is no summation over } t\text{)},$$
$$P^{in}_{s;\,in}=(2s+1),\ \sum_s P^{lm}_{s;\,ik}=\frac{1}{2}(\delta^l_i\delta^m_k+\delta^m_i\delta^l_k)\equiv\delta^{lm}_{ik}. \tag{7.2}$$

It is convenient to first write the operators P_s in the momentum representation. To this end we introduce the following auxiliary (projection) quantities:

$$X_{ik}=\frac{1}{\sqrt{3}}\left(\gamma_{ik}-\frac{q_iq_k}{q^2}\right),\quad Y_{ik}=\frac{q_iq_k}{q^2}. \tag{7.3}$$

It can easily be demonstrated that the operators P_s satisfying (7.2) can be written, via (7.3), in the following form:

$$P^{ml}_{0;\,ni}=X_{ni}X^{lm},\quad P^{ml}_{0';\,ni}=Y_{ni}Y^{ml}, \tag{7.4}$$

$$P^{ml}_{1;\,ni}=\frac{\sqrt{3}}{2}[X^l_iY^m_n+X^m_nY^l_i+X^m_iY^l_n+X^l_nY^m_i], \tag{7.5}$$

$$P^{ml}_{2;\,ni}=\frac{3}{2}[X^l_iX^m_n+X^m_iX^l_n]-X_{ni}X^{ml}. \tag{7.6}$$

Formulas (7.4)-(7.6) show that the $P^{ml}_{s,ni}$ are symmetric in the indices (ml) and (ni).

In the x-representation the projection operators P_s are nonlocal integrodifferential operators:

$$(P^{ml}_{s;\,ni}f^{ni})=\int d^4yP^{lm}_{s;\,ni}(x-y)f^{ni}(y).$$

The explicit expressions for $P^{lm}_{0;\,ni}(x)$ and $P^{ml}_{2;\,ni}(x)$ have the form

$$P^{lm}_{0;\,ni}(x)=\frac{1}{3}[\gamma^{lm}\gamma_{in}\delta(x)+(\gamma^{lm}\partial_i\partial_n+\gamma_{in}\partial^l\partial^m)D(x)+\partial_i\partial_n\partial^l\partial^m\Delta(x)], \tag{7.7}$$

$$\begin{aligned}P^{lm}_{2;\,ni}(x)=&\left(\delta^{lm}_{in}-\frac{1}{3}\gamma^{lm}\gamma_{in}\right)\delta(x)+\frac{2}{3}\partial^l\partial^m\partial_i\partial_n\Delta(x)\\&+\Big[\frac{1}{2}(\delta^l_i\partial^m\partial_n+\delta^m_n\partial^l\partial_i+\delta^l_n\partial^m\partial_i+\delta^m_i\partial^l\partial_n)\\&-\frac{1}{3}(\gamma^{lm}\partial_i\partial_n+\gamma_{in}\partial^l\partial^m)\Big]D(x).\end{aligned} \tag{7.8}$$

In both (7.7) and (7.8), $D(x)$ is the Green function of the wave equation

$$\Box D(x)=-\delta(x), \tag{7.9}$$

and

$$\Delta(x) = \int d^4y D(x-y) D(y).$$

We therefore have the equation

$$\Box\Delta(x) = -D(x). \tag{7.10}$$

Using (7.7)-(7.10) we can easily verify that the operators P_0 and P_2 are conserved, that is, obey the following identities:

$$\begin{aligned} \partial_l P^{lm}_{0;\,ni}(x) = \partial^n P^{lm}_{0;\,ni}(x) \equiv 0, \\ \partial_l P^{lm}_{2;\,ni}(x) = \partial^n P^{lm}_{2;\,ni}(x) \equiv 0. \end{aligned} \tag{7.11}$$

But the operators P_1 and $P_{0'}$ do not exhibit this property.

Expansion (7.1) implies that if the tensor field obeys the equation

$$\partial_l f^{lm} = 0, \tag{7.12}$$

it does not contain the representations with spins 1 and 0′. This means that such a tensor field describes only spins 2 and 0.

In view of (7.7) and (7.8) it can easily be verified that the operator

$$\begin{aligned} \Box(2P_0 - P_2)^{mn}_{il} = &-(\delta^{mn}_{il} - \gamma^{mn}\gamma_{il})\Box\delta(x) - (\gamma^{mn}\partial_i\partial_l + \gamma_{il}\partial^m\partial^n)\delta(x) \\ &+ \frac{1}{2}(\delta^n_i\partial^m\partial_l + \delta^m_l\partial^n\partial_i + \delta^n_l\partial^m\partial_i + \delta^m_i\partial^n\partial_l)\delta(x) \end{aligned} \tag{7.13}$$

is the only second-order operator that is local and conserved. Acting with this operator on the function $\varphi^{il} - (1/2)\gamma^{il}\varphi$, where $\varphi = \gamma_{pq}\varphi^{pq}$, and allowing for (7.7)-(7.10), we find that

$$\begin{aligned} \Psi^{mn} &= \int \Box_y \Big[2P_0(x-y) - P_2(x-y)\Big]^{mn}_{il} \Big[\varphi^{il}(y) - \frac{1}{2}\gamma^{il}\varphi(y)\Big] d^4y \\ &= \partial_k\partial_p[\gamma^{nk}\varphi^{pm} + \gamma^{mk}\varphi^{pn} - \gamma^{kp}\varphi^{mn} - \gamma^{mn}\varphi^{kp}]. \end{aligned} \tag{7.14}$$

The structure (7.14) for any symmetric tensor field is noteworthy in that it is local and linear, it contains only second-order derivatives, and it satisfies the conservation law, that is, the divergence of Ψ^{mn} is identically nil:

$$\partial_m \Psi^{mn} \equiv 0. \tag{7.15}$$

In what follows we will need structure (7.14) written in terms of the covariant derivatives of the metric-tensor density $\tilde{g}^{lm}$ with respect to the Minkowski metric:

$$J^{mn} = D_k D_p[\gamma^{np}\tilde{g}^{km} + \gamma^{pm}\tilde{g}^{kn} - \gamma^{kp}\tilde{g}^{mn} - \gamma^{mn}\tilde{g}^{kp}]. \tag{7.16}$$

From (7.16) it follows that

$$D_m J^{mn} \equiv 0, \tag{7.17}$$

which we will call the basic identity, since it plays a fundamental role in the construction of RTG.

Chapter 8. RTG Equations

Einstein declared that the metric tensor g^{ik} of the Riemann space-time characterizes the gravitational field in GR. This, however, was a profound delusion and it must be discarded, since it is impossible to place physical boundary conditions on the behavior of g^{ik} because their asymptotics depends on the choice of the spatial coordinate system. In this chapter we construct, within the framework of relativity theory and the geometrization principle, the relativistic equations for matter and gravitational field.

The relationship between the effective metric of the field Riemann space-time and the gravitational field can be chosen, by definition, to be

$$\tilde{g}^{ik} = \sqrt{-g}\, g^{ik} = \sqrt{-\gamma}\, \gamma^{ik} + \sqrt{-\gamma}\, \Phi^{ik}. \tag{8.1}$$

Hence, Riemannian geometry emerges here as a certain effective geometry, generated by the action of a physical gravitational field in the Minkowski space-time on matter. But for this construction of an effective Riemannian metric in the Minkowski space-time variables to have physical meaning, we must ensure that tha gravitational-field equations contain the Minkowski space-time metric γ^{ik}. In our theory the tensor Φ^{ik} is the field variable of the gravitational field, and the physical boundary conditions must be formulated for this variable. We will assume that the gravitational field in general has only spins 2 and 0. These physical restrictions, as shown in Chapter 7, lead in Galilean coordinates to the following four equations for the gravitational field:

$$\partial_i \Phi^{ik} = \partial_i \tilde{g}^{ik} = 0. \tag{8.2}$$

The Riemannian geometry of space-time is determined by fixing the metric-tensor field $g_{ik}(x)$ in a certain system of coordinate maps. Although de Donder, 1921, 1926, and Fock, 1939, 1957, 1959, used conditions of the (8.2) type in GR (they called them harmonic conditions), they were not able to show in which space-time variables these conditions must be written. Nevertheless, Fock, in describing problems of the island type, as much as considered harmonic conditions in terms of global Cartesian coordinates. But where did he find global Cartesian coordinates? They have no place in Riemannian geometry. Intuitively he made a correct move, but he could not comprehend its significance. If he had clearly understood that Eqs. (8.2) are valid only in an inertial reference frame, in Galilean coordinates of the Minkowski space-time, he could have arrived at the conception of a gravitational field as a physical tensor field in the Minkowski space-time. Fock focused especially on the importance of harmonic coordinate conditions for the solution of island problems. For instance, he wrote (Fock, 1959, p. 351):

> The above remarks concerning the privileged character of the harmonic system of coordinates should not be understood, in any case, as some kind of prohibition of the use of other coordinate systems. Nothing is more alien to our point of view than such an interpretation.

And further (p. 352):

> ...the existence of harmonic coordinates, ... though a fact of primary theoretical and practical importance, does not in any way preclude the use of other, non-harmonic, coordinate systems.

Fock also wrote (see Fock, 1939, p. 409):

> We believe that the possibility deserves to be noted of introducing, in the general theory of relativity, a fixed inertial coordinate system in a unique manner.

Developing this idea, Fock could probably have arrived at the concept of a gravitational field possessing energy-momentum density. But he did not. Did he attempt to consider the gravitational field as one of the Faraday-Maxwell type in the Minkowski space-time? No, he was far from this idea and explicitly said so (see Fock, 1939, p. 409):

> We mention this only in connection with the wish observed at times (which we in no way share) to place the theory of gravity into the framework of Euclidean space.

In GR, as Fock wrote (see Fock, 1959, p. 326),

> Gravitational energy can be separated out in the form of additional terms in the energy tensor only in an artificial manner by fixing the coordinate system and reformulating the problem in such a way that the gravitational field is taken to be superimposed on a space-time of fixed properties, just as is done in Newtonian theory. The additional terms in the energy tensor that correspond to gravitational energy do not possess the property of covariance (i.e. they do not form a tensor).

And further (p. 326):

> According to the choice of coordinate system the values of these terms at a given space-time point may prove to be zero or non-zero, which would be impossible for a tensor [this is still not understood by some researchers—*The authors*]. Therefore gravitational energy cannot be localized.

Irrespective of some insights, Fock was deeply averse to both the idea of the Minkowski space-time and the idea of the gravitational field of the Faraday-Maxwell type as playing any role in the theory of gravity. From the standpoint of our theory, Fock in solving island problems unconsciously dealt simply with ordinary Galilean coordinates in an inertial reference frame, and the latter, as is known from the theory of relativity, are preferred, of course. As a result, in his calculations involving island systems, the harmonic conditions emerged not as coordinate conditions, as he believed, but, as we will later see, as field equations in Galilean coordinates of an inertial reference frame.

Thus, Fock considered harmonic conditions only as preferred coordinate conditions and nothing more, and only for problems of the island type. This is understandable, since he, like all his great predecessors, was chained to Riemannian geometry, which in principle did not allow for a deeper penetration of the essence of the problem. To take this important step and advance these conditions as universal and covariant, it was necessary to repudiate the ideology of GR, get out of the jungle of Riemannian geometry, extend, contrary to the GR prescription, the principle of relativity to gravitational phenomena, and introduce the idea of a gravitational field as a physical field in the sense of Faraday and Maxwell, that is, possessing energy and momentum. All this has been done in our theory, with an arbitrary choice of system of coordinates, fixed only by the metric tensor γ^{ik} of the Minkowski space-time, as is common practice in the theory of elementary particles. Equations (8.2) are universal in our theory, since they are equations governing the gravitational field and have no relation to the choice of the coordinate systems. In the Minkowski space-time these equations can be written in covariant form as follows:

$$\sqrt{-\gamma}\, D_i\Phi^{ik} = D_i\tilde{g}^{ik} = 0. \tag{8.3}$$

Only in Cartesian (Galilean) coordinates do the field equations (8.3) assume the form of harmonic conditions. But writing the harmonic conditions within the GR framework in terms of Cartesian coordinates runs contrary to the GR ideology since in the Riemann space-time there can be no global Cartesian coordinates.

On the basis of Chapter 7 we can say that the field equations (8.3) automatically exclude spins 1 and 0′ from the gravitational tensor field. Thus, for the fourteen sought variables describing the gravitational field and matter we have already built four covariant equations (8.3). To construct the other ten, we use a simple but far reaching analogy with the electromagnetic field. Since any vector field A^n contains spins 1 and 0, it can be expanded in a direct sum of appropriate irreducible representations. This expansion can be realized via the projection operators (7.3) introduced in Chapter 7:

$$A^n = X^n_m A^m + Y^n_m A^m, \tag{8.4}$$

where operator X^n_m is conserved, that is, satisfies the identities

$$\partial_n X^n_m = \partial^n X^n_m \equiv 0, \tag{8.5}$$

while operator Y^n_m does not possess this property.

From electrodynamics it is known that the source of an electromagnetic field A^n is a conserved electromagnetic current j^n. Therefore, in constructing the equation of motion of the field it is natural to use the conserved operator X^n_m, too. This operator is nonlocal, but on its basis we can build a unique, local, linear, and conserved operator $\Box X^n_m$ that contains only second derivatives. Applying this operator to A^m, we get an expression which in terms of covariant derivatives has the form

$$\gamma^{mk} D_m D_k A^n - D^n D_m A^m.$$

Postulating the equation

$$\gamma^{mk} D_m D_k A^n - D^n D_m A^m = 4\pi j^n, \tag{8.6}$$

we arrive at the well-known Maxwell equations.

One of the most important features of the electrodynamics equation (8.6) is that it is invariant under the following gauge transformation:

$$A^n \to A^n + D^n \varphi, \tag{8.7}$$

with φ an arbitrary scalar function.

None of the physical quantities is affected by the gauge transformation (8.7). This means that none depends on the presence of spin 0 in the vector field A^n. Hence, the gauge transformation can be selected such that spin 0 would be excluded once and for all from the vector field. This means introducing the condition

$$D_m A^m = 0. \tag{8.8}$$

Thus, in electrodynamics condition (8.8) can be introduced, but this is not a necessary condition because spin 0 of the vector field has no effect on physical quantities due to gauge invariance.

Allowing for (8.8) in (8.6), we arrive at a system of equations

$$\gamma^{mk} D_m D_k A^n = 4\pi j^n, \tag{8.9a}$$

$$D_m A^m = 0, \tag{8.9b}$$

which determines a vector potential A^n possessing only spin 1.

The Lagrangian formalism that leads to these results is well known. Note that the idea of constructing a theory of interactions of vector fields (both Abelian and non-Abelian) based on gauge invariance proved to be extremely fruitful and is being successfully developed.

The problems that we encounter in setting up the remaining equations for a gravitational tensor field are of quite a different nature, since the source of this field, the energy-momentum tensor, is noninvariant under gauge transformations of field $\widetilde{\Phi}^{ik}$. We will discuss this aspect in greater detail later. For the present, by analogy with Maxwell's electrodynamics, we will construct the remaining equations for the gravitational tensor field. The second-rank tensor that is conserved is the energy-momentum tensor of matter and gravitational field in the Minkowski space-time, t^{mn}. Hence, it is natural to take it as the ultimate source of the gravitational field. Since, as established in Chapter 7, the simplest identically conserved tensor linear in $\widetilde{g}^{mn}$ is J^{mn}, by analogy with electrodynamics we can postulate the validity of the following equations:

$$J^{mn} \equiv D_k D_p [\gamma^{kn}\widetilde{g}^{pm} + \gamma^{km}\widetilde{g}^{pn} - \gamma^{kp}\widetilde{g}^{mn} - \gamma^{mn}\widetilde{g}^{kp}] = \lambda\,(t^{mn}_{(g)} + t^{mn}_{(M)}). \tag{8.10}$$

Generally speaking, such a type of equation presupposes the automatic validity of the law of conservation of the energy-momentum tensor of matter and gravitational field in the Minkowski space-time,

$$D_m\,(t^{mn}_{(g)} + t^{mn}_{(M)}) \equiv D_m t^{mn} = 0, \tag{8.11}$$

and, as a corollary (see Eq. (6.27)), the validity of the covariant conservation law for matter in the Riemann space-time:

$$\nabla_m T^{mn} = 0. \tag{8.12}$$

The Hilbert energy-momentum tensor T^{mn} can be specified phenomenologically. In this case Eqs. (8.12) constitute the equations of motion of matter.

Combining (8.3) with (8.10), we get

$$\gamma^{kp} D_k D_p \widetilde{g}^{mn} = -\lambda\,(t^{mn}_{(g)} + t^{mn}_{(M)}), \tag{8.13a}$$

$$D_m \widetilde{g}^{mn} = 0. \tag{8.13b}$$

This system of equations, (8.13a) and (8.13b), is the sought system for RTG.

The role of Eqs. (8.13b) in RTG is essentially different from the role that (8.8) plays in electrodynamics. Indeed, although the left-hand side of (8.10) is invariant under the gauge transformation

$$\widetilde{g}^{mn} \to \widetilde{g}^{mn} + D^m\widetilde{\xi}^n + D^n\widetilde{\xi}^m - \gamma^{mn} D_k \widetilde{\xi}^k, \tag{8.14}$$

where $\widetilde{\xi}^n = \sqrt{-\gamma}\,\xi^n$ is the density of an arbitrary 4-vector $\xi^n(x)$, in the theory we do not have the arbitrariness of the (8.14) type since the right-hand side of (8.10) is noninvariant under transformation (8.14). For this reason, Eqs. (8.3) cannot follow from Eqs. (8.10).

Hence, in RTG Eqs. (8.3) constitute additional independent dynamical equations for the gravitational field rather than coordinate conditions.

The main problem in constructing a theory is to establish whether there exists a Lagrangian density for a gravitational field with spins 2 and 0 that would automatically lead, via the principle of least action, to Eqs. (8.13a). The total Lagrangian density of a gravitational field $\widetilde{\Phi}^{ik}$ that describes spins 2 and 0 and is quadratic in the first derivatives of the field has the form

$$\begin{aligned} L_g = {} & a\widetilde{g}_{km}\widetilde{g}_{nq}\widetilde{g}^{lp} D_l \widetilde{g}^{kq} D_p \widetilde{g}^{mn} + b\widetilde{g}_{kq} D_m \widetilde{g}^{pq} D_p \widetilde{g}^{km} \\ & + c\widetilde{g}_{km}\widetilde{g}_{nq}\widetilde{g}^{lp} D_l \widetilde{g}^{km} D_p \widetilde{g}^{nq}. \end{aligned} \tag{8.15}$$

A characteristic feature of this Lagrangian is that the convolution of covariant derivatives taken with respect to the Minkowski metric is achieved via the effective metric tensor $\widetilde{g}^{ik}$ of the Riemann space-time. It can be shown that this restriction on the gravitational field is a consequence of the geometrization principle and the structure of the gravitational field, which possesses spins 2 and 0.

In view of the principle of least action, the system of equations for the gravitational field assumes the form

$$\frac{\delta L_g}{\delta \widetilde{\Phi}^{ik}} + \frac{\delta L_M}{\delta \widetilde{\Phi}^{ik}} = \frac{\delta L_g}{\delta \widetilde{g}^{ik}} + \frac{\delta L_M}{\delta \widetilde{g}^{ik}} = 0. \tag{8.16}$$

where we have allowed for (8.1), L_M is the material Lagrangian density, and L_g is specified in (8.15).

To represent the system of equations (8.16) in the form (8.13a) we must select in an unambiguous manner the constants a, b, and c in the Lagrangian density (8.15). To this end we use formulas (6.17), (6.22), and (6.25) and find for Lagrangian $L = L_g + L_M$ the energy-momentum tensor density t^{mn} for matter and gravitational field in the Minkowski space-time. Calculating the variation of the total Lagrangian over γ_{mn}, we find that

$$\begin{aligned} t^{mn} = {} & 2\sqrt{-\gamma}\left(\gamma^{nk}\gamma^{mp} - \frac{1}{2}\gamma^{mn}\gamma^{pk}\right)\frac{\delta L}{\delta \widetilde{g}^{kp}} + 2bJ^{mn} \\ & + D_p\{(2a+b)\,[H^{pn}_k\gamma^{km} + H^{pm}_k\gamma^{kn} - H^{mn}_k\gamma^{kp}] \\ & - 2(a+2c)\,\gamma^{mn}\widetilde{g}^{kp}\widetilde{g}_{lq}D_k\widetilde{g}^{lq}\}, \end{aligned} \tag{8.17}$$

where

$$H^{pn}_k = (\widetilde{g}^{pl}D_l\widetilde{g}^{qn} + \widetilde{g}^{nl}D_l\widetilde{g}^{pq})\,\widetilde{g}_{qk}.$$

We see that the equations

$$\begin{aligned} t^{mn} = {} & 2bJ^{mn} + D_p\{(2a+b)\,[H^{pn}_k\gamma^{km} + H^{pm}_k\gamma^{kn} - H^{mn}_k\gamma^{kp}] \\ & - 2(a+2c)\,\gamma^{mn}\widetilde{g}^{kp}\widetilde{g}_{lq}D_k\widetilde{g}^{lq}\} \end{aligned} \tag{8.18}$$

are equivalent to the field equations (8.16). If we wish the condition

$$D_m t^{mn} = 0 \tag{8.19}$$

not to produce any new equation for field Φ^{ik}, since this would lead to an overdetermined system of equations, it is necessary and sufficient that the coefficients a, b, and c satisfy the following conditions:

$$a = -\frac{1}{2}b, \quad c = \frac{1}{4}b. \tag{8.20}$$

If the constants are selected in this manner, we arrive at an identity:

$$D_m t^{mn} \equiv 0.$$

Thus, the equations of the motion of matter follow directly from the equations for the gravitational field. Allowing for (8.20), we find that (8.18) assumes the form

$$\begin{aligned} D_pD_k(\gamma^{km}\widetilde{g}^{pn} + \gamma^{kn}\widetilde{g}^{pm} - \widetilde{g}^{mn}\gamma^{kp} - \gamma^{mn}\widetilde{g}^{kp}) & = \frac{1}{2b}\left(t^{mn}_{(g)} + t^{mn}_{(M)}\right) \\ & = \frac{1}{2b}t^{mn}. \end{aligned} \tag{8.21}$$

This coincides with Eqs. (8.10), which were written by analogy with electrodynamics, if we put $2b = 1/\lambda$.

Thus, the Lagrangian density that leads us to field equations in the form of (8.21) is

$$L_g = \frac{1}{2\lambda}\left[\tilde{g}_{kq} D_m \tilde{g}^{lq} D_p \tilde{g}^{km} - \frac{1}{2}\tilde{g}_{km}\tilde{g}_{nq}\tilde{g}^{lp} D_l \tilde{g}^{kq} D_p \tilde{g}^{mn} + \frac{1}{4}\tilde{g}_{km}\tilde{g}_{nq}\tilde{g}^{lp} D_l \tilde{g}^{km} D_p \tilde{g}^{nq}\right]. \tag{8.22}$$

The correspondence principle implies that

$$\lambda = -16\pi. \tag{8.23}$$

If we allow for (8.23) in (8.22), we get

$$L_g = \frac{1}{32\pi}\left[\tilde{G}^l_{mn} D_l \tilde{g}^{mn} - \tilde{g}^{mn}\tilde{G}^k_{mk}\tilde{G}^l_{nl}\right], \tag{8.24}$$

where the third-rank tensor $\tilde{G}^k_{lm}$ is defined thus:

$$\tilde{G}^k_{lm} = \frac{1}{2}\tilde{g}^{pk}\left(D_m \tilde{g}_{lp} + D_l \tilde{g}_{mp} - D_p \tilde{g}_{lm}\right). \tag{8.25}$$

We can also write L_g in the form

$$L_g = -\frac{1}{16\pi}\sqrt{-g}\, g^{mn}\left[G^k_{lm} G^l_{nk} - G^l_{mn} G^k_{lk}\right]. \tag{8.26}$$

The first to consider such a Lagrangian was Rosen, 1940, 1963. The third-rank tensor G^k_{ml} in (8.26) is defined as follows:

$$G^k_{ml} = \frac{1}{2} g^{kp}\left(D_m g_{pl} + D_l g_{pm} - D_p g_{lm}\right). \tag{8.27}$$

It is easily verified that Lagrangian (8.26) can be transformed into the sum of two terms, one of which does not contain the metric coefficients γ^{mn} and the other, which depends on γ^{mn}, is written in the form of the divergence of a vector and, therefore, does not affect the field equations.

If we allow for Eq. (8.3), the complete system of RTG equations for matter and gravitational field is (see Logunov and Mestvirishvili, 1984, 1985a, 1985b, 1986b, Vlasov and Logunov, 1984, and Vlasov, Logunov, and Mestivirishvili, 1984)

$$\gamma^{pk} D_p D_k \tilde{g}^{mn} = 16\pi t^{mn}, \tag{8.28}$$

$$D_m \tilde{g}^{mn} = 0. \tag{8.29}$$

Obviously, in a Galilean system of coordinates Eqs. (8.28), (8.29) assume the form

$$\Box\, \tilde{g}^{mn} = 16\pi t^{mn}, \tag{8.28'}$$

$$\partial_m \tilde{g}^{mn} = 0. \tag{8.29'}$$

Equations (8.28) and (8.29) clearly show that the Minkowski space-time enters into all the gravitational-field equations in an essential way. But this means that it will find its physical reflection not only in the fundamental laws of nature but also in the description of various natural phenomena.

The general-covariant RTG equations (8.28) and (8.29) closely resemble the general-covariant equations of electrodynamics, (8.9a) and (8.9b), in the absence of gravitational fields. In electrodynamics the electromagnetic field is a vector

field and its source is the conserved electromagnetic current $j^m(x)$. Equation (8.9b) excludes spin 0 from the vector field. In RTG the gravitational field is a tensor field and the source is the conserved tensor density of the energy-momentum of both matter and gravitational field. For this reason Eq. (8.28) is nonlinear even for a free gravitational field. Equation (8.29) excludes spins 1 and 0′ from the tensor field.

The equations of RTG and electrodynamics acquire an especially simple form in the Galilean coordinates in an inertial reference frame.

If we were to restrict our discussion to the first system of equations (8.28), the division of the metric of the Riemann space-time into the metric in the Minkowski space-time and the gravitational tensor field would be of a purely nominal nature and with no physical meaning. The second system (8.29) of four field equations drastically separates everything that refers to forces of inertia from everything that refers to the gravitational field. The two systems of equations, (8.28) and (8.29), are general covariant. The behavior of the gravitational field is restricted, as usual, by appropriate physical conditions in a given, say Galilean, system of coordinates. In GR it is impossible to formulate the physical conditions imposed on metric g^{mn} if one remains within the framework of the Riemann space-time, since the asymptotic behavior of the metric always depends on the choice of the three-dimensional system of coordinates.

Let us now find the explicit form of the system of equations (8.16). If we take Lagrangian (8.22), it can be demonstrated that

$$\frac{\partial L_g}{\partial \tilde{g}^{mn}} = \frac{1}{16\pi}\left[G^k_{ml}G^l_{kn} - G^k_{mn}G^l_{kl}\right]$$

and

$$\frac{\partial L_g}{\partial (D_k\tilde{g}^{mn})} = \frac{1}{16\pi}\left[G^k_{mn} - \frac{1}{2}\,\delta^k_m G^l_{nl} - \frac{1}{2}\,\delta^k_n G^l_{ml}\right].$$

Hence,

$$\frac{\delta L_g}{\delta \tilde{g}^{mn}} \equiv \frac{\partial L_g}{\partial \tilde{g}^{mn}} - D_k\,\frac{\partial L_g}{\partial (D_k\tilde{g}^{mn})} = -\frac{1}{16\pi}\,R_{mn}, \tag{8.30}$$

where R_{mn} is the second-rank tensor of the curvature of the Riemann space-time:

$$R_{mn} = D_k G^k_{mn} - D_m G^l_{nl} + G^k_{mn}G^l_{kl} - G^k_{ml}G^l_{nk}. \tag{8.31}$$

Since in view of (6.6b) and (6.11) we have

$$2\,\frac{\delta L_M}{\delta \tilde{g}^{mn}} = \frac{1}{\sqrt{-g}}\left(T_{mn} - \frac{1}{2}\,g_{mn}T\right), \tag{8.32}$$

Eq. (8.16) yields

$$\sqrt{-g}\;R_{mn} = 8\pi\left(T_{mn} - \frac{1}{2}\,g_{mn}T\right). \tag{8.33}$$

that is, we have arrived at the system of Hilbert-Einstein equations, the one important difference being that all field variables in the Hilbert-Einstein equations in our theory depend on universal spatial-temporal coordinates in the Minkowski space-time. In an inertial reference frame these universal coordinates can be chosen to be Galilean. It must be emphasized that the system of equations (8.28) does not coincide with the system of Hilbert-Einstein equations (8.33). Only if the general-covariant equations (8.29) hold true does the system of Hilbert-Einstein equations, formally written in GR in the variables of the Minkowski space-

time, reduce to the system of equations (8.28), and these depend essentially on the metric tensor of the Minkowski space-time.

It has long been known (see Rosen, 1940, 1963, and Tolman, 1934) that Lagrangian (8.26) leads to system (8.33). We have shown, however, that for a gravitational field with spins 2 and 0 the gravitational-field Lagrangian density (8.22) is the only one that leads to a self-consistent system of equations for matter and field, (8.28) and (8.29). This means that the RTG equations are the only simplest second-order equations that can exist.

In view of the importance of the equivalence of Eqs. (8.28) and (8.33) in the Minkowski variables, we can give another variant of the proof of the above statement based on direct calculations of the tensor densities $t^{mn}_{(g)}$ and $t^{mn}_{(M)}$, provided that (8.29) is valid.

If we take formulas (6.17) and the Lagrangian density (8.22) and allow for (8.1), we will find that the gravitational-field energy-momentum tensor density in the Minkowski space-time is

$$t^{mn}_{(g)} = -\frac{1}{16\pi} J^{mn} + \frac{\sqrt{-\gamma}}{8\pi}\left(\gamma^{mp}\gamma^{nk} - \frac{1}{2}\gamma^{mn}\gamma^{pk}\right) R_{pk}. \qquad (8.34)$$

We see that the second-rank curvature tensor R_{pk} of the Riemann space-time has emerged automatically. Similarly, using formulas (6.17) and (8.1) and the definition (6.6a) of the Hilbert-tensor density, we arrive at the following formula for the material energy-momentum tensor density in the Minkowski space-time:

$$t^{mn}_{(M)} = (\gamma/g)^{1/2}\left(\gamma^{mp}\gamma^{nk} - \frac{1}{2}\gamma^{mn}\gamma^{pk}\right)\left(T_{pk} - \frac{1}{2} g_{pk} T\right). \qquad (8.35)$$

Substituting (8.34) and (8.35) into the field equations (8.10), we get

$$\left(\gamma^{mp}\gamma^{nk} - \frac{1}{2}\gamma^{nm}\gamma^{pk}\right)\left[R_{pk} - \frac{8\pi}{\sqrt{-g}}\left(T_{pk} - \frac{1}{2} g_{pk} T\right)\right] = 0,$$

which leads us to the system of equations for the gravitational field in the form of (8.33).

The complete system of equations for matter and gravitational field, (8.28) and (8.29), is equivalent to the following system of equations:

$$\sqrt{-g}\, R_{mn} = 8\pi\left(T_{mn} - \frac{1}{2} g_{mn} T\right), \qquad (8.36)$$

$$D_m \tilde{g}^{mn} = 0. \qquad (8.37)$$

Thus, although in RTG the complete system of equations (8.36) and (8.37) does contain the system of Hilbert-Einstein equations, the content of the latter changes substantially,* since the spatial-temporal variables now coincide with the variables of the Minkowski space-time. We must again emphasize that Eqs. (8.37) are universal, since they are field equations describing gravitational fields with spins 2 and 0; they unambiguously separate forces of inertia from gravitational fields. Within the framework of GR this is impossible to do in principle. The choice of the reference frame (or system of coordinates) is fixed by the metric tensor of the Minkowski space-time, while Eqs. (8.37) lay no restrictions on the choice of the coordinate system.

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* Equations (8.36) do not contain metric γ_{ik}, and it is meaningless to speak of γ_{ik} in GR. This implies that the statement of Zel'dovich and Grishchuk, 1986, that GR can be constructed on the basis of the Minkowski space-time is erroneous.

Note that some aspects of the theory of gravitation in the Minkowski space-time have been considered in Gupta, 1952, Köhler, 1952, 1953, 1954, Papapetrou, 1948, Pugachev, 1958, 1959, 1964, Rosen, 1940, 1963, and Thirring, 1961. However, even scientists who were on the right track at the beginning failed to understand this and took a different direction in building the theory of gravitation, a direction that has not led to a complete theory.

In conclusion, one remark is in order. The system (8.3) whose validity we have postulated, does not follow from the principle of least action. Therefore, in applying this principle to Lagrangian (8.15), we were forced to allow for Eqs. (8.3) by introducing in the integrand in the action integral a term of the form $\eta_m D_n \tilde{g}^{mn}$, where η_m are Lagrange's multipliers. An analysis of this problem can be found in Appendix 2.

Chapter 9. Relationships Between the Canonical Energy-Momentum Tensor and the Hilbert Tensor

The gravitational-field Lagrangian density depends on the metric-tensor density $\tilde{\gamma}^{mn}$, the gravitational-field tensor density $\tilde{\Phi}^{mn}$, and their first derivatives. Under a coordinate transformation (6.1) the variation of action, δJ_g, is zero:

$$\delta J_g = \int_\Omega d^4x \left[D_k J^k + \frac{\delta L_g}{\delta \tilde{\Phi}^{mn}} \delta_L \tilde{\Phi}^{mn} + \frac{\delta L_g}{\delta \tilde{\gamma}^{mn}} \delta_L \tilde{\gamma}^{mn} \right] = 0. \tag{9.1}$$

Here

$$J^k = -\xi^p \tau_p^k + K_m^{pk} D_p \xi^m, \tag{9.2}$$

where the canonical tensor density

$$\tau_p^k = -\delta_p^k L_g + \frac{\partial L_g}{\partial (\partial_k \tilde{\Phi}^{mn})} D_p \tilde{g}^{mn} = -\delta_p^k L_g + \frac{\partial L_g}{\partial (D_k \tilde{g}^{mn})} D_p \tilde{g}^{mn}, \tag{9.3}$$

and the third-rank tensor density K_m^{pk} has the form

$$K_m^{pk} = 2 \frac{\partial L_g}{\partial (\partial_k \tilde{\Phi}^{mn})} \tilde{\Phi}^{pn} - \delta_m^p \frac{\partial L_g}{\partial (\partial_k \tilde{\Phi}^{lq})} \tilde{\Phi}^{lq} + 2 \frac{\partial L_g}{\partial (\partial_k \tilde{\gamma}^{mn})} \tilde{\gamma}^{pn} - \delta_m^p \frac{\partial L_g}{\partial (\partial_k \tilde{\gamma}^{lq})} \tilde{\gamma}^{lq}. \tag{9.4}$$

Substituting into (9.1) formulas (6.14) and (6.15) for variations $\delta_L \tilde{\Phi}^{mn}$ and $\delta_L \tilde{\gamma}^{mn}$ and bearing in mind that the integration volume Ω is arbitrary, we obtain the following identity:

$$\xi^p \left[D_k \tau_p^k + \frac{\delta L_g}{\delta \tilde{\Phi}^{mn}} D_p \tilde{\Phi}^{mn} \right] - K_m^{pk} D_p D_k \xi^m + D_p \xi^m \left[\tau_m^p - D_k K_m^{pk} - 2 \frac{\delta L_g}{\delta \tilde{\Phi}^{mn}} \tilde{\Phi}^{pn} + \delta_m^p \frac{\delta L_g}{\delta \tilde{\Phi}^{ql}} \tilde{\Phi}^{ql} - 2 \frac{\delta L_g}{\delta \tilde{\gamma}^{mn}} \tilde{\gamma}^{pn} + \delta_m^p \frac{\delta L_g}{\delta \tilde{\gamma}^{ql}} \tilde{\gamma}^{ql} \right] = 0. \tag{9.5}$$

Since the displacement vector ξ^p is arbitrary, the last expression yields the following strong identities:

$$D_k \tau_p^k = -\frac{\delta L_g}{\delta \widetilde{\Phi}^{mn}} D_p \widetilde{\Phi}^{mn}, \tag{9.6}$$

$$\tau_m^k - D_p K_m^{kp} = 2\frac{\delta L_g}{\delta \widetilde{\Phi}^{mn}} \widetilde{\Phi}^{kn} - \delta_m^k \frac{\delta L_g}{\delta \widetilde{\Phi}^{ql}} \widetilde{\Phi}^{ql} + 2\frac{\delta L_g}{\delta \widetilde{\gamma}^{mn}} \widetilde{\gamma}^{kn} - \delta_m^k \frac{\delta L_g}{\delta \widetilde{\gamma}^{ql}} \widetilde{\gamma}^{ql}, \tag{9.7}$$

$$K_m^{kp} = -K_m^{pk}. \tag{9.8}$$

Since the metric-tensor density $\widetilde{g}^{mn}$ of the effective Riemann space-time and the gravitational-field tensor density $\widetilde{\Phi}^{mn}$ are linked by the relationship

$$\widetilde{g}^{mn} = \widetilde{\gamma}^{mn} + \widetilde{\Phi}^{mn}, \tag{9.9}$$

we can write

$$\frac{\delta L_g}{\delta \widetilde{\Phi}^{mn}} = \frac{\delta L_g}{\delta \widetilde{g}^{mn}}, \quad \frac{\partial L_g}{\partial (\partial_p \widetilde{\Phi}^{mn})} = \frac{\partial L_g}{\partial (D_p \widetilde{g}^{mn})}.$$

With these equations in mind, we can write

$$\frac{\partial L_g}{\partial (\partial_p \widetilde{\gamma}^{mn})} = \frac{\partial L_g}{\partial (D_p \widetilde{g}^{mn})} - \widetilde{g}^{qj} \frac{\partial L_g}{\partial (D_k \widetilde{g}^{qj})} \frac{\partial \gamma_{kl}^l}{\partial (\partial_p \widetilde{\gamma}^{mn})} + \widetilde{g}^{qj} \frac{\partial L_g}{\partial (D_k \widetilde{g}^{lj})} \frac{\partial \gamma_{qk}^l}{\partial (\partial_p \widetilde{\gamma}^{mn})}.$$

Here the γ_{qk}^l are Cristoffel symbols for the Minkowski space-time:

$$\gamma_{qk}^l = \frac{1}{2} \gamma^{lj} (\partial_q \gamma_{kj} + \partial_k \gamma_{jq} - \partial_j \gamma_{qk}).$$

Elementary calculations lead us to the following expression for K_m^{kp}:

$$K_m^{kp} = \frac{\partial L_g}{\partial (D_p \widetilde{g}^{mn})} \widetilde{g}^{kn} - \frac{\partial L_g}{\partial (D_k \widetilde{g}^{mn})} \widetilde{g}^{pn} + \widetilde{g}^{qn} \widetilde{\gamma}_{qm} \left[\frac{\partial L_g}{\partial (D_k \widetilde{g}^{ln})} \widetilde{\gamma}^{pl} - \frac{\partial L_g}{\partial (D_p \widetilde{g}^{ln})} \widetilde{\gamma}^{kl} \right] + \frac{\partial L_g}{\partial (D_l \widetilde{g}^{qn})} \widetilde{\gamma}_{lm} [\widetilde{g}^{kn} \widetilde{\gamma}^{pq} - \widetilde{g}^{pn} \widetilde{\gamma}^{kq}]. \tag{9.10}$$

Since the energy-momentum tensor density of the gravitational field is defined as

$$t_{(g)m}^k = 2\frac{\delta L_g}{\delta \widetilde{\gamma}^{mn}} \widetilde{\gamma}^{kn} - \delta_m^k \frac{\delta L_g}{\delta \widetilde{\gamma}^{pq}} \widetilde{\gamma}^{pq}, \tag{9.11}$$

we can write (9.7) as follows:

$$t_{(g)m}^k = \tau_m^k - D_p K_m^{kp} - 2\frac{\delta L_g}{\delta \widetilde{\Phi}^{mn}} \widetilde{\Phi}^{kn} + \delta_m^k \frac{\delta L_g}{\delta \widetilde{\Phi}^{pq}} \widetilde{\Phi}^{pq}. \tag{9.12}$$

This form establishes a relationship between the Hilbert-tensor density in the Minkowski space-time and the canonical energy-momentum tensor density.

For future discussions it is expedient to introduce the following quantity as a characteristic of the gravitational field:

$$\overset{(0)}{t}{}^{k}_{(g)m} = \tau^k_m - D_p K^{kp}_m, \tag{9.13}$$

which in the case of a free gravitational field coincides, in view of (9.12), with the Hilbert energy-momentum tensor density.

The system of equations (8.28), (8.29) for matter and gravitational field can be written in a somewhat different form in terms of the Hilbert energy-momentum tensor density in the Riemann space-time. To this end let us calculate the third-rank tensor density K^{pk}_m using formula (9.10) and the Lagrangian given by (8.24) and (8.25).

Employing the fact that

$$\frac{\partial L_g}{\partial (D_k \tilde{g}^{mn})} = \frac{1}{16\pi}\left[\tilde{G}^k_{mn} + \frac{1}{2}\tilde{g}^{kp}\tilde{g}_{mn}\tilde{G}^l_{lp}\right],$$

we obtain

$$\begin{aligned}16\pi K^{pk}_m = [\tilde{g}^{pn}\tilde{G}^k_{mn} - \tilde{g}^{kn}\tilde{G}^p_{mn}] + \tilde{g}^{nq}\tilde{\gamma}_{qm}[\tilde{\gamma}^{kl}\tilde{G}^p_{ln} - \tilde{\gamma}^{pl}\tilde{G}^k_{ln}] \\ + \tilde{\gamma}_{ml}\tilde{G}^l_{qn}[\tilde{g}^{pn}\tilde{\gamma}^{kq} - \tilde{g}^{kn}\tilde{\gamma}^{pq}].\end{aligned}$$

Substituting the expression for $\tilde{G}^k_{mn}$ (8.25), we find that

$$\begin{aligned}16\pi K^{pk}_m = \tilde{g}_{mn} D_q(\tilde{g}^{kq}\tilde{g}^{pn} - \tilde{g}^{pq}\tilde{g}^{kn}) \\ - \gamma_{mn} D_q(\tilde{g}^{kq}\gamma^{pn} + \tilde{g}^{pn}\gamma^{kq} - \tilde{g}^{pq}\gamma^{kn} - \tilde{g}^{kn}\gamma^{pq}).\end{aligned} \tag{9.14}$$

Combining this with the definition (9.13) of $\overset{(0)}{t}{}^{k}_{(g)m}$, we arrive at the following formula:

$$\overset{(0)}{t}{}^{k}_{(g)m} = \tau^k_m - \frac{1}{16\pi} D_p \sigma^{kp}_m - \frac{1}{16\pi}\gamma_{nm} J^{kn}, \tag{9.15}$$

where the antisymmetric tensor density σ^{kp}_m is defined as

$$\sigma^{kp}_m = \tilde{g}_{mn} D_q(\tilde{g}^{pq}\tilde{g}^{kn} - \tilde{g}^{kq}\tilde{g}^{pn}), \tag{9.16}$$

and by J^{kn} we denote the well-known quantity (7.16).

Substituting into (9.12) the expression (9.14) for K^{pk}_m and going on to variations in metric $\tilde{g}^{mn}$, we obtain

$$\begin{aligned}t^k_{(g)m} = \tau^k_m - \frac{1}{16\pi} D_p \sigma^{kp}_m - \frac{1}{16\pi}\gamma_{mn} J^{kn} + \frac{\sqrt{-g}}{8\pi}\left[R^k_m - \frac{1}{2}\delta^k_m R\right] \\ - \frac{\sqrt{-\gamma}}{8\pi}\left[R_{mn}\gamma^{kn} - \frac{1}{2}\delta^k_m R_{pq}\gamma^{pq}\right].\end{aligned} \tag{9.17}$$

Now if we use formula (8.34) and lower one index via metric γ_{nm}, we find that

$$t^k_{(g)m} = -\frac{1}{16\pi}\gamma_{mn} J^{kn} - \frac{\sqrt{-\gamma}}{8\pi}\left[R_{mn}\gamma^{kn} - \frac{1}{2}\delta^k_m R_{pq}\gamma^{pq}\right]. \tag{9.18}$$

Comparing (9.17) and (9.18), we arrive at the following relationship:

$$\tau^k_m - \frac{1}{16\pi} D_p \sigma^{kp}_m = -\frac{\sqrt{-g}}{8\pi}\left[R^k_m - \frac{1}{2}\delta^k_m R\right]. \tag{9.19}$$

Here the canonical tensor density τ^k_m is given by (9.3), that is,

$$\tau^k_m = -\delta^k_m L_g + \frac{\partial L_g}{\partial (D_k \tilde{g}^{pq})} D_m \tilde{g}^{pq},$$

where the Lagrangian density L_g is written in terms of covariant derivatives in the Minkowski metric and has the form (8.24). Empolying (9.19), we can write (9.15) as

$$\overset{(0)}{t}{}^k_{(g)m} = -\frac{\sqrt{-g}}{8\pi}\left[R^k_m - \frac{1}{2}\delta^k_m R\right] - \frac{1}{16\pi}\gamma_{mn}J^{nk}. \tag{9.20}$$

Earlier we established that the RTG system of equations (8.28), (8.29) for matter and gravitational field is equivalent to the system of equations (8.36), (8.37). Equation (9.20) can be used to write the RTG system of equations for matter and gravitational field in an equivalent form, namely,

$$\gamma_{ml}\gamma^{pq}D_p D_q \tilde{g}^{nl} = 16\pi\,(T^n_m + \overset{(0)}{t}{}^n_{(g)m}), \tag{9.21}$$

$$D_m \tilde{g}^{mn} = 0, \tag{9.22}$$

where T^n_m is the Hilbert energy-momentum tensor density (6.6a) for matter in the Riemann space-time. Obviously, in view of Eqs. (9.21) and (9.22), the law of conservation of the energy-momentum tensor density for matter and gravitational field has the form

$$D_n\,(T^n_m + \overset{(0)}{t}{}^n_{(g)m}) = D_n\,(t^n_{(g)m} + t^n_{(M)m}) = D_n\,(T^n_m + \tau^n_m) = 0. \tag{9.23}$$

The covariant law of matter conservation in the Riemann space-time can be represented in equivalent form:

$$\nabla_n T^n_m = \partial_n T^n_m - \frac{1}{2}T^{nq}\partial_m g_{nq} \equiv D_n T^n_m - G^q_{mn}T^n_q = 0. \tag{9.24}$$

Comparison of (9.23) and (9.24) yields

$$G^q_{mn}T^n_q = -D_n \overset{(0)}{t}{}^n_{(g)m}. \tag{9.25}$$

The last expression clearly shows that matter receives energy and momentum directly from the gravitational field, with the total energy-momentum tensor of matter and gravitational field being conserved exactly. Construction of the relativistic theory of gravitation on the basis of the Minkowski space-time and the geometrization principle enabled us to deal only with covariant quantities at each stage of our discussion.

In conclusion a remark is in order. Using (9.19), we can write the RTG Hilbert-Einstein equations in the form

$$D_p \sigma^{kp}_m = (T^k_m + \tau^k_m)\,16\pi, \tag{9.26}$$

where σ^{kp}_m is given by (9.16) and constitutes a third-rank tensor, and τ^k_m, whose explicit form for Lagrangian (8.24) is

$$\begin{aligned}\tau^k_m = \frac{1}{32\pi}\Big[\delta^k_m\Big(\tilde{g}_{lq}D_j\tilde{g}^{pq}D_p\tilde{g}^{lj} + \frac{1}{2}\tilde{g}^{lp}D_l\tilde{g}_{nj}D_p\tilde{g}^{jn} + \frac{1}{4}\tilde{g}_{ij}\tilde{g}_{nq}\tilde{g}^{lp}D_l\tilde{g}^{ij}D_p\tilde{g}^{nq}\Big)\\ -\Big(2\tilde{g}_{np}D_q\tilde{g}^{nk} + \tilde{g}^{kn}D_n\tilde{g}_{pq} + \frac{1}{2}\tilde{g}_{pq}\tilde{g}_{ln}\tilde{g}^{kj}D_j\tilde{g}^{nl}\Big)D_m\tilde{g}^{pq}\Big],\end{aligned} \tag{9.27}$$

is a second-rank tensor with respect to general coordinate transformations. The tensor quantities τ^k_m and σ^{kp}_m contain covariant derivatives with respect to the

Minkowski space-time metric γ^{ik}, but in the difference $\tau^k_m - D_p\sigma^{kp}_m$ the dependence on the metric tensor γ^{ik} cancels out if we ignore the RTG equations (8.3) (see Appendix 1). Hence, the Hilbert-Einstein equations assume the form

$$\partial_p\sigma^{kp}_m = (T^k_m + \tau^k_m)\,16\pi. \tag{9.28}$$

Here neither τ^k_m nor $\partial_p\sigma^{kp}_m$ contains γ^{ik} and the two have common partial derivatives and, hence, are not tensors with respect to general coordinate transformations.

Thus, if we ignore the RTG equations (8.3), the Hilbert-Einstein equations (8.33) can be represented only in form (9.28), since the introduction of metric γ^{ik} in GR has no physical meaning. The Hilbert-Einstein equations written in terms of arbitrary coordinates of the Riemann space-time lead to a differential conservation law of the form

$$\partial_k(T^k_m + \tau^k_m) = 0. \tag{9.29}$$

But this law has no physical meaning, a fact that can easily be verified if for the variables x^k we take spherical coordinates, for example. Relation (9.29) can be given physical meaning if we consider it in Cartesian (or Galilean) coordinates. However, in the Riemann space-time, and therefore in GR, it is impossible to introduce global Cartesian coordinates.

RTG equations have meaning only in the coordinates of the Minkowski space-time, and there is no way in which these equations can be rid of the metric tensor of this space-time. The conservation law here assumes the form

$$D_k(T^k_n + \tau^k_n) = 0, \tag{9.30}$$

with τ^k_m a tensor (see (9.27)). Selecting Cartesian coordinates leads to the following law of conservation:

$$\partial_k(T^k_m + \tau^k_m) = 0. \tag{9.31}$$

In RTG, just as in other physical theories, the metric γ^{ik} enters into the field equations (9.22) and fixes a certain chosen class of reference frames. In this class the law of conservation is uniquely defined and has the covariant form (9.30). In RTG, in Galilean coordinates of the Minkowski space-time, the law of conservation acquires the especially simple form (9.31) and, as usual, the choice of these coordinates does not violate the covariance of the theory.

Since Eqs. (8.3) contain the metric tensor of the Minkowski space-time, γ^{ik}, by substituting these equations into (9.26) we can reduce the latter to equations that contain γ^{ik}. The solution of RTG equations depends on the choice of the metric of the Minkowski space-time.

Chapter 10. The Gauge Principle and the Uniqueness of the RTG Lagrangian

When in Chapter 8 we considered the Lagrangian density (8.15) of the gravitational field $\tilde{\Phi}^{ik}$, a Lagrangian density describing spins 2 and 0 (Eq. (8.15) does not contain covariant derivatives of the form $D_m\tilde{g}^{mk}$) and quadratic in the first derivatives $D_m\tilde{\Phi}^{ik} \equiv D_m\tilde{g}^{ik}$, we arrived at the unambiguous conclusion that only a Lagrangian of the form (8.24) leads us to field equations of the (8.21) type.

In this chapter, to construct the RTG Lagrangian density L_g for a free gravitational field, we employ a method quite different from the one used in Chapter 8. We will use the local gauge group of transformations of the gravitational field and formulate the gauge principle.

Let us consider the Hilbert-Einstein equation in the weak-field approximation. For $\tilde{\Phi}^{mn}$ in the first order in the gravitational constant G we have the following formula:

$$\tilde{\Phi}^{mn} = \sqrt{-\gamma}\overset{(0)}{\Phi}{}^{mn} - \frac{1}{2}\tilde{\gamma}^{mn}\overset{(0)}{\Phi}{}^{k}_{k},$$

where $\overset{(0)}{\Phi}{}^{mn}$ is a solution to the linearized equation (8.36). As is known (Thirring, 1961), the linearized Hilbert-Einstein equation is invariant under a local gauge transformation

$$\overset{(0)}{\Phi}{}^{mn} \to \overset{(0)}{\Phi}{}^{mn} + \gamma^{mk}D_k\varepsilon^n(x) + \gamma^{nk}D_k\varepsilon^m(x),$$

where the 4-vector $\varepsilon^n(x)$ is an arbitrary infinitesimal parameter. When $\overset{(0)}{\Phi}{}^{mn}$ is transformed in this manner, the infinitesimal increments of $\tilde{\Phi}^{mn}$ and $\tilde{g}^{mn}$ in an approximation linear in G have the form

$$\delta\tilde{\Phi}^{mn} \equiv \delta\tilde{g}^{mn} = \tilde{\gamma}^{mk}D_k\varepsilon^n + \tilde{\gamma}^{nk}D_k\varepsilon^m - D_k(\varepsilon^k\tilde{\gamma}^{mn}). \tag{10.1}$$

For an arbitrary gravitational field a natural generalization of the local gauge transformation is

$$\delta_\varepsilon\tilde{\Phi}^{mn} \equiv \delta_\varepsilon\tilde{g}^{mn} = \tilde{g}^{mk}\nabla_k\varepsilon^n + \tilde{g}^{nk}\nabla_k\varepsilon^m - \nabla_k(\varepsilon^k\tilde{g}^{mn}). \tag{10.2}$$

This expression can be obtained from (10.1) if on the right-hand side of (10.1) we add the gravitational field $\tilde{\Phi}^{mn}$ directly to the metric-tensor density $\tilde{\gamma}^{mn}$, that is, if a stretching transformation

$$\tilde{\gamma}^{mn} \to \tilde{\gamma}^{mn} + \tilde{\Phi}^{mn} = \tilde{g}^{mn}$$

is performed. Since

$$\nabla_k\varepsilon^n = D_k\varepsilon^n + G^n_{kp}\varepsilon^p \tag{10.3}$$

and

$$\nabla_k(\varepsilon^k\tilde{g}^{mn}) = D_k(\varepsilon^k\tilde{g}^{mn}) + G^n_{kp}\varepsilon^k\tilde{g}^{pm} + G^m_{kp}\varepsilon^k\tilde{g}^{np}, \tag{10.4}$$

the right-hand side of (10.2) can be written in the equivalent form

$$\delta_\varepsilon\tilde{\Phi}^{mn} = \delta_\varepsilon\tilde{g}^{mn} = \tilde{g}^{mk}D_k\varepsilon^n + \tilde{g}^{nk}D_k\varepsilon^m - D_k(\varepsilon^k\tilde{g}^{mn}). \tag{10.5}$$

Similarly, on the basis of

$$D_k\varepsilon^n = \partial_k\varepsilon^n + \gamma^n_{kp}\varepsilon^p \tag{10.6}$$

and

$$D_k(\varepsilon^k\tilde{g}^{mn}) = \partial_k(\varepsilon^k\tilde{g}^{mn}) + \gamma^n_{kp}\varepsilon^k\tilde{g}^{pm} + \gamma^m_{kp}\varepsilon^k\tilde{g}^{np}, \tag{10.7}$$

we obtain for (10.5) the following representation:

$$\delta_\varepsilon\tilde{\Phi}^{mn} = \delta_\varepsilon\tilde{g}^{mn} = \tilde{g}^{mk}\partial_k\varepsilon^n + \tilde{g}^{nk}\partial_k\varepsilon^m - \partial_k(\varepsilon^k\tilde{g}^{mn}). \tag{10.8}$$

Comparing (10.2), (10.5), and (10.8), we see that the right-hand sides are invariant under the transformations $\nabla_k \leftrightarrow D_k$, $D_k \leftrightarrow \partial_k$, and $\nabla_k \leftrightarrow \partial_k$.

Let us now demonstrate that the operators δ_ε form a Lie algebra. To this end we use representation (10.5), according to which

$$\delta_{\varepsilon_2}(\delta_{\varepsilon_1}\tilde{g}^{mn}) = \delta_{\varepsilon_1}\tilde{g}^{mk}D_k\varepsilon_2^n + \delta_{\varepsilon_1}\tilde{g}^{nk}D_k\varepsilon_2^m - \varepsilon_2^k D_k(\delta_{\varepsilon_1}\tilde{g}^{mn}) - D_k\varepsilon_2^k\delta_{\varepsilon_1}\tilde{g}^{mn},$$

$$\delta_{\varepsilon_1}(\delta_{\varepsilon_2}\tilde{g}^{mn}) = \delta_{\varepsilon_2}\tilde{g}^{mk}D_k\varepsilon_1^n + \delta_{\varepsilon_2}\tilde{g}^{nk}D_k\varepsilon_1^m - \varepsilon_1^k D_k(\delta_{\varepsilon_2}\tilde{g}^{mn}) - D_k\varepsilon_1^k\delta_{\varepsilon_2}\tilde{g}^{mn}.$$

Hence, for the Lie commutator we have the following formula:

$$(\delta_{\varepsilon_2}\delta_{\varepsilon_1} - \delta_{\varepsilon_1}\delta_{\varepsilon_2})\,\tilde{g}^{mn} = \delta_{\varepsilon_3}\tilde{g}^{mn}, \tag{10.9}$$

where

$$\varepsilon_3^l = \varepsilon_1^k D_k\varepsilon_2^l - \varepsilon_2^k D_k\varepsilon_1^l. \tag{10.10}$$

Using Eqs. (10.3) and (10.6), we can easily demonstrate that the (10.10) is form-invariant under the transformations $\partial_k \leftrightarrow D_k$, $\partial_k \leftrightarrow \nabla_k$, and $D_k \leftrightarrow \nabla_k$.

Note that while for $\delta_\varepsilon\tilde{g}^{mn}$ Eq. (10.5) formally coincides with Eq. (6.3) for an infinitesimal increment of $\tilde{g}^{mn}(x)$ generated by coordinate transformations (6.1), for field $\tilde{\Phi}^{mn}(x)$ Eq. (10.5) differs substantially from an infinitesimal increment of the field, (6.14), generated by coordinate transformations (6.1). Hence, gauge transformations (10.5) are supercoordinate.

We are now ready to formulate the gauge principle. If the Lagrangian density changes under a gauge transformation (10.5) only by the divergence of a function, we will say that it satisfies the gauge principle. It is easy to show that under a gauge transformation, that is,

$$\tilde{g}^{mn}(x) \to \tilde{g}^{mn}(x) + \delta_\varepsilon\tilde{g}^{mn}(x), \tag{10.11}$$

with $\delta_\varepsilon\tilde{g}^{mn}(x)$ specified by (10.5), Lagrangian (8.22) is transformed according to the law

$$L_g \to L_g + D_k Q^k(x), \tag{10.12}$$

with

$$Q^k(x) = -\varepsilon^k L_g - \frac{1}{\lambda}\left[D_m\varepsilon^n D_n\tilde{g}^{mk} + \varepsilon^k D_m D_n\tilde{g}^{mn} - D_n(\varepsilon^n D_m\tilde{g}^{mk})\right]. \tag{10.13}$$

Hence, the free-gravitational-field Lagrangian density (8.22) found in Chapter 8 satisfies the gauge principle.

Let us now establish the general requirement imposed on the structure of a Lagrangian density that satisfies the gauge principle. Suppose that L_g changes under a (10.5) transformation only by the divergence of a function. Then the variation of action for such an L_g is zero:

$$\delta_\varepsilon J_g = \int \delta_\varepsilon L_g(\tilde{\gamma}^{mn},\ \tilde{g}^{mn},\ D_k\tilde{g}^{mn})\, d^4x = 0. \tag{10.14}$$

In the first order in $\varepsilon^l(x)$, $D_k\varepsilon^l(x)$, and $D_m D_k\varepsilon^l(x)$ we have

$$\delta_\varepsilon L_g = \frac{\partial L_g}{\partial\tilde{g}^{mn}}\,\delta_\varepsilon\tilde{g}^{mn} + \frac{\partial L_g}{\partial(D_k\tilde{g}^{mn})}\,D_k\delta_\varepsilon\tilde{g}^{mn}(x). \tag{10.15}$$

Substituting (10.15) into (10.14) and integrating by parts, we obtain

$$\int \frac{\delta L_g}{\delta\tilde{g}^{mn}}\,\delta_\varepsilon\tilde{g}^{mn} d^4x = 0. \tag{10.16}$$

Taking into account Eq. (10.5) for $\delta_\varepsilon \tilde{g}^{mn}$ and integrating (10.16) by parts, we get

$$\int d^4x\, \varepsilon^k(x) \left[D_l \left(2 \frac{\delta L_g}{\delta \tilde{g}^{ik}} \tilde{g}^{il} \right) - D_k \left(\frac{\delta L_g}{\delta \tilde{g}^{il}} \right) \tilde{g}^{il} \right] = 0. \tag{10.17}$$

In view of the arbitrariness of $\varepsilon^k(x)$ we have

$$D_l \left(2 \frac{\delta L_g}{\delta \tilde{g}^{ik}} \tilde{g}^{il} \right) - D_k \left(\frac{\delta L_g}{\delta \tilde{g}^{il}} \right) \tilde{g}^{il} \equiv 0. \tag{10.18}$$

Since

$$D_l \left(2 \frac{\delta L_g}{\delta \tilde{g}^{ik}} \tilde{g}^{il} \right) - \tilde{g}^{il} D_k \left(\frac{\delta L_g}{\delta \tilde{g}^{il}} \right) = \partial_l \left(2 \frac{\delta L_g}{\delta \tilde{g}^{ik}} \tilde{g}^{il} \right) - \tilde{g}^{il} \partial_k \left(\frac{\delta L_g}{\delta \tilde{g}^{il}} \right),$$

we can rewrite identity (10.18) as follows:

$$D_l \left(2 \frac{\delta L_g}{\delta \tilde{g}^{ik}} \tilde{g}^{il} \right) - \tilde{g}^{il} D_k \left(\frac{\delta L_g}{\delta \tilde{g}^{il}} \right) = -\tilde{g}_{ik} \nabla_l \left(2 \frac{\delta L_g}{\delta \tilde{g}_{il}} \right) \equiv 0, \tag{10.19}$$

where, as usual, ∇_l is the symbol for the covariant derivative with respect to metric g_{ik}.

Identity (10.19) constitutes the requirement imposed on the structure of a gravitational-field Lagrangian density L_g satisfying the gauge principle. Clearly, any scalar density depending only on g_{ik} and/or the partial derivatives of g_{ik} satisfies (10.19). The simplest examples are

$$L_g = \sqrt{-g} \tag{10.20}$$

and

$$L_g = \sqrt{-g}\, R, \tag{10.21}$$

where R is the scalar curvature of the Riemann space-time. Direct substitution verifies that under a transformation of the (10.5) type expressions (10.20) and (10.21) transform, respectively, according to the following laws:

$$\sqrt{-g} \to \sqrt{-g} - D_l (\varepsilon^l \sqrt{-g}), \tag{10.22}$$

$$\sqrt{-g}\, R \to \sqrt{-g}\, R - D_l (\varepsilon^l \sqrt{-g}\, R). \tag{10.23}$$

Hence, the scalar densities (10.20) and (10.21) satisfy the gauge principle.

However, as noted in Chapter 6, the choice of a Lagrangian density depending only on g^{ik} and/or the partial derivatives of g^{ik} does not, in view of (6.29), satisfy our initial requirements, since in this case the gravitational field is not of the Faraday-Maxwell type. Hence, in accordance with our concept, it is expedient to construct a Lagrangian density that is not completely geometrized, that is, depends on both g^{ik} and γ^{ik} and on the first derivatives of g^{ik} and γ^{ik}. It has been established that such a solution exists and is unique. (This will be proved in Appendix 3.) Here we will employ the transformation law (10.23) and construct a Lagrangian density that is not completely geometrized. It will prove to be the RTG Lagrangian density (8.24).

We can easily show that the Riemann-Cristoffel curvature tensor can be expressed as follows (see Appendix 1):

$$R^n_{mpq} = D_q G^n_{mp} - D_p G^n_{mq} + G^l_{mp} G^n_{lq} - G^l_{mq} G^n_{lp}, \tag{10.24}$$

where G^k_{mn} is given by (8.27), and D_m stands, as usual, for the operator of covariant differentiation with respect to the Minkowski metric.

Contraction of (10.24) on n and q leads to the Ricci tensor

$$R_{mp}=R^{n}_{mpn}=D_nG^n_{mp}-D_pG^n_{mn}+G^l_{mp}G^n_{ln}-G^l_{mn}G^n_{lp}. \tag{10.25}$$

Multiplying $\tilde{g}^{mp}=\sqrt{-g}\,g^{mp}$ into (10.25) and summing over indices m and p, we find the expression for the scalar curvature density $\tilde{R}$:

$$\begin{aligned}\tilde{R}&=\sqrt{-g}\,R=\sqrt{-g}\,g^{mp}R_{mp}\\&=\tilde{g}^{mp}(D_nG^n_{mp}-D_pG^n_{mn})+\tilde{g}^{mp}(G^l_{mp}G^n_{ln}-G^l_{mn}G^n_{lp}).\end{aligned} \tag{10.26}$$

It is easy to verify that

$$\tilde{g}^{mp}(D_nG^n_{mp}-D_pG^n_{mn})=D_n(\tilde{g}^{mp}G^n_{mp}-\tilde{g}^{mn}G^p_{mp})-2\tilde{g}^{mp}[G^l_{mp}G^n_{ln}-G^l_{mp}G^n_{lp}].$$

Combining this with (10.26) yields

$$\tilde{R}=D_n(\tilde{g}^{mp}G^n_{mp}-\tilde{g}^{mn}G^p_{mp})-\tilde{g}^{mp}[G^l_{mp}G^n_{ln}-G^l_{mn}G^n_{lp}]. \tag{10.27}$$

Note that each group of terms on the right-hand side of (10.27),

$$D_n(\tilde{g}^{mp}G^n_{mp}-\tilde{g}^{mn}G^p_{mp}) \tag{10.28}$$

and

$$\tilde{g}^{mp}[G^l_{mp}G^n_{ln}-G^l_{mn}G^n_{lp}], \tag{10.29}$$

constitutes a scalar density, since under a general coordinate transformation the two transform separately as scalar densities. Moreover, in contrast to $\tilde{R}$, the expressions (10.28) and (10.29) are not completely geometrized, since they explicitly depend on γ^{mn} and the partial derivatives of γ^{mn}.

Since under a gauge transformation (10.5) $\tilde{R}$ defined by (10.27) changes according to (10.23), the scalar Lagrangian density of the form

$$L_g=\frac{1}{\lambda}(\tilde{R}-D_nS^n), \tag{10.30}$$

where S^n is a vector density constructed from $\tilde{g}^{mn}$ and $D_k g^{mn}$, satisfies the gauge principle. Note that L_g (10.30) is also not completely geometrized because D_nS^n is not completely geometrized either.

If we now require that the gravitational-field Lagrangian density be quadratic in the first derivatives $D_p g^{mn}$, then necessarily

$$S^n=\tilde{g}^{mp}G^n_{mp}-\tilde{g}^{mn}G^p_{mp}.$$

The right-hand side of (10.30) yields (10.29) multiplied by $-1/\lambda$, and if we put $\lambda=16\pi$, we arrive at the RTG Lagrangian density (8.26).

We note once more that the RTG Lagrangian desity (8.26) is a scalar Lagrangian density for arbitrary coordinate transformation, a condition that GR cannot meet.

A characteristic feature of both the Lagrangian density (8.15) and the Lagrangian density (10.30) is that the convolution of the covariant derivatives $D_m\tilde{g}^{kl}$ in these densities can be carried out only by applying the effective metric tensor $\tilde{g}^{ml}$, a condition that can be explained by the special way in which the gravitational field acts upon itself.

In Appendix 3 we will consider the general form of the Lagrangian density, quadratic in the first derivatives $D_m\tilde{g}^{kl}$ (including the terms that contain $D_m\tilde{g}^{mk}$), with the convolution carried out via the metric tensor $\tilde{\gamma}^{mn}$ of the Minkowski space-time. There we will show that the gauge principle unambiguously reduces the Lagrangian density to (8.26).

In the presence of matter the RTG equations do not admit gauge transformations (10.5). In this respect these gauge transformations differ from those used in electrodynamics, which are valid for the case of interacting fields, too. In the absence of matter the gauge transformations (10.5) do not affect the gravitational field equations but change the line element in the Riemann space-time and, hence, the geometric characteristics of this space-time. It can easily be verified that

$$\begin{gathered}\delta_\varepsilon ds^2 = \delta_\varepsilon g_{ik}\,dx^i\,dx^k,\\ \delta_\varepsilon R_{ik} = -R_{il}D_k\varepsilon^l - R_{kl}D_i\varepsilon^l - \varepsilon^l D_l R_{ik},\\ \delta_\varepsilon R_{iklm} = -R_{qklm}D_i\varepsilon^q - R_{iqlm}D_k\varepsilon^q - R_{ikqm}D_l\varepsilon^q - R_{iklq}D_m\varepsilon^q - \varepsilon^q D_q R_{iklm}.\end{gathered} \tag{10.31}$$

Here one can see the difference between transformations (10.5) and gauge invariance of electrodynamics, where gauge transformations do not affect the physical observables. The geometry of space-time is uniquely determined in the presence of matter, since in this case gauge arbitrariness is absent. When no matter is present, the RTG metric tensor of the effective Riemann space-time is fixed only after nonphysical components of the gravitational field are eliminated by applying gauge supercoordinate transformations.

In our theory the equations of motion of matter follow from the ten equations for the gravitational field $\tilde{\Phi}^{ik}$. Thus, for ten variables of the gravitational field $\tilde{\Phi}^{ik}$ and the four variables characterizing matter we have only ten equations. The choice of the coordinate system in RTG is specified completely by the metric tensor γ^{ik} of the Minkowski space-time. For the system of equations to be complete we need four more covariant field equations. Such equations were introduced in Chapter 8 as equations that determine the structure of the gravitational field of the Faraday-Maxwell type with spins 2 and 0. These equations, specifically Eqs. (8.3), are universal. When applied to a free gravitational field, they restrict the class of possible gauge transformations by imposing on $\varepsilon^l(x)$ the condition

$$g^{mn}D_mD_n\varepsilon^l(x) = 0. \tag{10.32}$$

Thus, the combination of the gauge principle and the idea of a gravitational field as a physical field of the Faraday-Maxwell type possessing energy, momentum, and spins 2 and 0 unambiguously leads us to the system of equations of RTG.

Note that for a static and spherically symmetric gravitational field Eq. (10.32) can have only a zero solution; hence, the gauge arbitrariness is completely lifted in this case.

Chapter 11. A Generalization of the RTG System of Equations

In this chapter we approach the problem of constructing the RTG system of equations from another angle. As before, we assume that the gravitational field is a tensor field containing irreducible representations, which correspond to states with spins 2 and 0. There are four equations that determine such a structure of the gravitational field, and they have the form (8.3). The total number of variables

of the gravitational field and matter are 15. The material variables are the three components of velocity, energy density, and pressure, while the gravitational field is characterized by the ten components of the symmetric tensor density Φ^{mn} Introducing equations of state, we can find the relationship between the energy density and the pressure. Thus, to construct a complete system of equations, we need ten more equations in addition to the four equations (8.3).

To set up the free-gravitational-field Lagrangian density we will employ the gauge principle formulated as follows: a Lagrangian density is said to satisfy the gauge principle if under gauge transformations (10.1) it changes by a divergence on a class of vectors $\varepsilon^l(x)$ satisfying condition (10.32).

Allowing for properties (10.22) and (10.23), we can write the free-gravitational-field Lagrangian density satisfying the gauge principle in the following general form:

$$L_g = \frac{1}{16\pi}(\tilde{R} - D_k S^k) - \frac{1}{16\pi}\left[\Lambda\sqrt{-g} + \frac{1}{2}m^2\gamma_{ik}\tilde{g}^{ik} + \varkappa_0\sqrt{-\gamma}\right], \qquad (11.1)$$

where we have taken into account Eqs. (8.3). To exclude in the Lagrangian density terms containing second derivatives, we select the vector density S^k as follows:

$$S^k = \tilde{g}^{mp}G^k_{mp} - \tilde{g}^{km}G^p_{mp}. \qquad (11.2)$$

Combining (10.27) and (11.2) yields the following formula for the free-gravitational-field Lagrangian density:

$$L_g = -\frac{1}{16\pi}\tilde{g}^{ik}[G^p_{ik}G^n_{pn} - G^n_{ip}G^p_{kn}] - \frac{1}{16\pi}\left[\Lambda\sqrt{-g} + \frac{1}{2}m^2\gamma_{ik}\tilde{g}^{ik} + \varkappa_0\sqrt{-\gamma}\right]. \qquad (11.3)$$

The Lagrangian density of gravitational field and matter is

$$L = L_g + L_M. \qquad (11.4)$$

On the basis of (11.4), allowing for (8.1), we can calculate the energy-momentum density of matter and gravitational field, t^{mn}, in the Minkowski space-time. It has the form

$$t^{mn} = 2\sqrt{-\gamma}\left(\gamma^{nk}\gamma^{mp} - \frac{1}{2}\gamma^{mn}\gamma^{pk}\right)\frac{\delta L}{\delta\tilde{g}^{pk}} - \frac{1}{16\pi}J^{mn} + \frac{1}{16\pi}[m^2\tilde{g}^{mn} + \varkappa_0\tilde{\gamma}^{mn}], \qquad (11.5)$$

where J^{nm} is defined in (7.16).

In view of the principle of least action

$$\frac{\delta L}{\delta\tilde{g}^{pk}} = 0, \qquad (11.6)$$

we have the following system of equations:

$$R^{mn} + \frac{1}{2}\Lambda g^{mn} + \frac{1}{2}m^2 g^{mk}g^{np}\gamma_{kp} = \frac{8\pi}{\sqrt{-g}}\left(T^{mn} - \frac{1}{2}g^{mn}T\right). \qquad (11.7)$$

Since in the absence of matter ($T^{mn} = 0$) and gravitational field ($\Phi^{mn} = 0$) Eqs. (11.7) must be satisfied automatically, we have

$$\Lambda = -m^2. \qquad (11.8)$$

Thus, the complete system of equations for matter and gravitational field has the form:

$$R^{mn}-\frac{1}{2}m^2[g^{mn}-g^{mk}g^{np}\gamma_{kp}]=\frac{8\pi}{\sqrt{-g}}\left(T^{mn}-\frac{1}{2}g^{mn}T\right), \quad (11.9)$$

$$D_m\tilde{g}^{mn}=0. \quad (11.10)$$

We see that the metric tensor of the Minkowski space-time enters into both Eq. (11.9) and (11.10). Equations (11.10) are clearly necessary since only their presence ensures that the law of energy-momentum conservation for matter and gravitational field combined is valid, which means that Eqs. (11.10) are in no way related to the gauge conditions. The system of equations (11.9)-(11.10) will be complete at $m = 0$, too.

Since in the absence of a field Φ^{mn} the gravitational-field energy-momentum tensor density $t^{mn}_{(g)}$ must be zero, Eq. (11.5) yields

$$\varkappa_0 = -m^2. \quad (11.11)$$

Combining (11.5) with (11.6) and (11.11), we can represent the system of equations (11.9) and (11.10) in equivalent form

$$J^{mn}-m^2\tilde{\Phi}^{mn}=-16\pi t^{mn}, \quad (11.12)$$

$$D_m\tilde{\Phi}^{mn}=0. \quad (11.13)$$

If we include (11.13) in (11.12), we get

$$\gamma^{ik}D_iD_k\tilde{\Phi}^{mn}+m^2\tilde{\Phi}^{mn}=16\pi t^{mn}, \quad (11.14)$$

$$D_m\tilde{\Phi}^{mn}=0. \quad (11.15)$$

Note that Eqs. (11.14), (11.15) contain the metric tensor of the Minkowski space-time. Equations (11.9), as well as Eqs. (11.14), contain an unknown constant parameter m^2, whose physical meaning will be discussed later.

Let us now show that Eqs. (11.10) are necessary and sufficient for the covariant law of conservation of the energy-momentum tensor density of matter to be valid

$$\nabla_m T^{mn} = 0. \quad (11.16)$$

Let us write (11.9) in the form

$$\sqrt{-g}\left(R^{kl}-\frac{1}{2}g^{kl}R\right)+\frac{m^2}{2}\sqrt{-g}\left[g^{kl}+\left(g^{kp}g^{lq}-\frac{1}{2}g^{kl}g^{pq}\right)\gamma_{pq}\right] = 8\pi T^{kl}. \quad (11.17)$$

Since $\nabla_k g^{pq}\equiv 0$, $\nabla_k\sqrt{-g}\equiv 0$, and $\nabla_k\left(R^{kl}-\frac{1}{2}g^{kl}R\right)\equiv 0$, Eq. (11.17) yields

$$m^2\sqrt{-g}\left(g^{kp}g^{lq}-\frac{1}{2}g^{kl}g^{pq}\right)\nabla_k\gamma_{pq}-16\pi\nabla_k T^{kl}=0. \quad (11.18)$$

Consider the term in (11.18) that contains $\nabla_k\gamma_{pq}$. Since $\nabla_k\gamma_{pq}=-G^l_{kp}\gamma_{lq}-G^l_{kq}\gamma_{lp}$, with G^l_{kp} specified in (8.27), we obtain

$$\left(g^{kp}g^{lq}-\frac{1}{2}g^{kl}g^{pq}\right)\nabla_k\gamma_{pq}=\gamma_{kn}g^{kl}\left(D_pg^{pn}+G^q_{pq}g^{pn}\right). \quad (11.19)$$

If we combine

$$\sqrt{-g}\left(D_pg^{pn}+G^q_{pq}g^{pn}\right)\equiv D_p\tilde{g}^{pn} \quad (11.20)$$

with (11.18), (11.19), and (11.20), we get

$$m^2\gamma_{kn}g^{kl}D_p\tilde{g}^{pn}-16\pi\nabla_k T^{kl}=0. \tag{11.21}$$

We see that the covariant conservation law (11.16) is a direct corollary of the system of equations (11.9), (11.10). The following assertion is also true. If (11.16) is valid, then (11.21), which is a corollary of (11.9), necessarily implies the system of equations (11.10). Hence, the theory describes a massive gravitational field possessing only spins 2 and 0, since the system of equations (11.10) excludes the possibility of irreducible representations corresponding to spins 1 and 0′.

Applying D_m to (11.12) and allowing for the identity $D_m J^{mn} \equiv 0$, we obtain

$$m^2 D_m\tilde{\Phi}^{mn}=16\pi D_m t^{mn}. \tag{11.22}$$

This equation shows that the covariant law of conservation of the total energy-momentum tensor density (of matter and gravitational field taken together) in the Minkowski space-time,

$$D_m t^{mn}=0, \tag{11.23}$$

is a corollary of the system of equations (11.13). This conservation law and the system of equations (11.12) nec essarily lead to Eqs. (11.13).

What is the physical meaning of the parameter m^2? Consider the system of equations (11.14) in the weak-field approximation (Φ^{mn} is the weak field). In Galilean coordinates, relationship (8.1) yields the following expansions for g^{mn} and g:

$$g^{mn}=\gamma^{mn}+\Phi^{mn}-\frac{1}{2}\gamma^{mn}\Phi_k^k-\frac{1}{2}\Phi^{mn}\Phi_k^k$$
$$+\frac{1}{4}\gamma^{mn}\left(\Phi_{pq}\Phi^{pq}+\frac{1}{2}\Phi_p^p\Phi_q^q\right)+\dots,$$
$$g=-1-\Phi_k^k+\frac{1}{2}\Phi_{pq}\Phi^{pq}-\frac{1}{2}\Phi_p^p\Phi_q^q+\dots .$$

These expansions yield

$$R^{mn}\simeq\frac{1}{2}\left(\Box\Phi^{mn}-\frac{1}{2}\gamma^{mn}\Box\Phi_q^q\right),$$

and hence

$$\sqrt{-g}\left(R^{mn}-\frac{1}{2}g^{mn}R\right)\simeq\frac{1}{2}\Box\Phi^{mn}. \tag{11.24}$$

In the first order in field Φ^{mn} we have

$$\frac{1}{2}m^2\tilde{g}^{mn}+\frac{1}{2}m^2\sqrt{-g}\left(g^{mp}g^{nk}-\frac{1}{2}g^{mn}g^{pk}\right)\gamma_{pk}\simeq\frac{1}{2}m^2\Phi^{mn}.$$

Thus, in the weak-field approximation Eq. (11.17) assumes the form

$$(\Box+m^2)\Phi^{mn}=0. \tag{11.25}$$

We see that for a weak gravitational field constant m is the graviton mass.

Equations (11.9) are gauge noninvariant even in the absence of matter, $T^{mn}=0$. This means that the introduction of a mass term lifts the degeneracy with respect to gauge supercoordinate transformations, although the free-gravitational-field Lagrangian density (11.3) satisfies the gauge principle formulated in this chapter. The presence of a mass term makes it possible to unambiguously determine the geometry of space-time and the gravitational-field energy-momentum density in the absence of matter. In view of the fact that the mass term lifts the

degeneracy, its introduction may be considered a technical trick used in calculations, where in the final expression the term is nullified. This approach automatically leads to a physical solution. The formal passage in Eqs. (11.9), (11.10) or Eqs. (11.14), (11.15) to mass m equal to zero leads us, respectively, to RTG equations in the form (8.36), (8.37) or (8.28), (8.29). In this case, the metric tensor of the effective Riemann space-time outside matter has a gauge supercoordinate arbitrariness (10.5), with vectors $\varepsilon^l(x)$ satisfying Eq. (10.32). Since the main object of investigation in our theory is gravitational field, the gauge supercoordinate arbitrariness can be used to exclude the nonphysical components of the field. The metric tensor of the effective Riemann space-time must be constructed via (8.1) using only the physical components of the gravitational field. All this will be illustrated in Chapter 15 where we consider gravitational waves. If the action-at-a-distance principle can be applied to gravitational field, the mass term vanishes and the cosmological constant is absent.

Chapter 12. Solution of RTG Equations

12.1 The Field of a Spherically Symmetric Object

In this chapter, following Fock, 1939, 1959, Belinfante, 1955, Belinfante and Garrison, 1962, Tolman, 1934, Vlasov and Logunov, 1985a, 1986b, and Weinberg, 1972, we consider the solution of RTG equations for a spherically symmetric object in the case of a massless gravitational field.

We select the coordinates in the Minkowski space-time in the form

$$t, \quad x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\theta. \tag{12.1}$$

In what follows it will be expedient to employ the following notation for the coordinates t, r, θ, and φ:

$$t = x^0, \quad r = x^1, \quad \theta = x^2, \quad \varphi = x^3. \tag{12.2}$$

The metric coefficients in terms of these coordinates are

$$\begin{gathered}\gamma_{00} = 1, \ \gamma_{11} = -1, \quad \gamma_{22} = -r^2, \quad \gamma_{33} = -r^2\sin^2\theta;\\ \gamma^{00} = 1, \quad \gamma^{11} = -1, \quad \gamma^{22} = -1/r^2, \quad \gamma^{33} = -1/r^2\sin^2\theta;\\ \gamma_{mn} = \gamma^{mn} = 0 \text{ for } m \neq n, \quad \sqrt{-\gamma} = r^2\sin\theta.\end{gathered} \tag{12.3}$$

The nonzero Christoffel symbols are

$$\begin{gathered}\gamma^1_{22} = -r, \quad \gamma^1_{33} = -r\sin^2\theta, \quad \gamma^2_{12} = \gamma^3_{13} = r^{-1},\\ \gamma^2_{33} = -\sin\theta\cos\theta, \quad \gamma^3_{23} = \cot\theta.\end{gathered} \tag{12.4}$$

In formulas (12.3) and (12.4) and in what follows the number subscripts and superscripts stand for the spherical coordinates according to (12.2).

To solve the RTG equations means to construct an effective Riemannian manifold, that is, find the metric tensor $g^{ik}(x)$ of the Riemann space-time.

Since the Hilbert-Einstein equations are included in the RTG system of equations, we will focus our attention on these equations. These will enable us in the future to determine the significance of the field general-covariant equations (8.37). Such an approach will result in certain conclusions concerning GR.

Let us now find the field generated by a static and spherically symmetric source. Here the general form of the line element (Landau and Lifshitz, 1975, Tolman, 1934, and Weinberg, 1972) is

$$ds^2 = g_{00}dt^2 + 2\,g_{01}dt\,dr + g_{11}dr^2 + g_{22}d\theta^2 + g_{33}d\varphi^2, \tag{12.5}$$

where the metric coefficients g_{00}, g_{01}, g_{11}, and g_{22} are functions only of the radial variable r, while g_{33} depends on r and angle θ. Let us introduce the following notation:

$$\begin{gathered} g_{00}(r) = U(r), \quad g_{01}(r) = -A(r), \quad g_{11}(r) = -\left[V(r) - \frac{A^2(r)}{U(r)}\right], \\ g_{22}(r) = -W(r), \quad g_{33}(r,\theta) = -W(r)\sin^2\theta. \end{gathered} \tag{12.6}$$

It can be demonstrated that the nonzero components of tensor g^{mn} have the form

$$\begin{gathered} g^{00}(r) = \frac{1}{U}\left(1 - \frac{A^2}{UV}\right), \quad g^{01}(r) = -\frac{A}{UV}, \quad g^{11}(r) = -\frac{1}{V}, \\ g^{22}(r) = -\frac{1}{W}, \quad g^{33}(r,\theta) = -\frac{1}{W\sin^2\theta}, \end{gathered} \tag{12.7}$$

and the determinant of the metric tensor g_{mn} is given as follows:

$$g = \det g_{mn} = -UVW^2\sin^2\theta. \tag{12.8}$$

The functions $U(r)$, $V(r)$, $A(r)$, and $W(r)$ must be found from the RTG equations (8.36), (8.37). Let us write the left-hand side of Eq. (8.36) in terms of U, V, A, and W. We start by finding the nonzero components of the connection coefficient

$$\Gamma^l_{mn} = \frac{1}{2}g^{lk}(\partial_m g_{kn} + \partial_n g_{km} - \partial_k g_{mn}). \tag{12.9}$$

Employing (12.6), (12.7), and (12.9), we obtain

$$\begin{gathered} \Gamma^0_{00} = \frac{A}{2UV}\frac{\partial U}{\partial r}, \quad \Gamma^0_{01} = \frac{1}{2U}\frac{\partial U}{\partial r}\left(1 - \frac{A^2}{UV}\right), \\ \Gamma^0_{11} = -\frac{1}{U}\frac{\partial A}{\partial r} + \frac{A}{2UV}\frac{\partial V}{\partial r} + \frac{A^3}{2U^3V}\frac{\partial U}{\partial r}, \\ \Gamma^0_{22} = -\frac{A}{2UV}\frac{\partial W}{\partial r}, \quad \Gamma^0_{33} = \sin^2\theta\,\Gamma^0_{22}, \\ \Gamma^1_{00} = \frac{1}{2V}\frac{\partial U}{\partial r}, \quad \Gamma^1_{01} = -\frac{A}{2UV}\frac{\partial U}{\partial r}, \\ \Gamma^1_{11} = \frac{1}{2V}\frac{\partial V}{\partial r} + \frac{A^2}{2U^2V}\frac{\partial U}{\partial r}, \quad \Gamma^1_{22} = -\frac{1}{2V}\frac{\partial W}{\partial r}, \\ \Gamma^1_{33} = \sin^2\theta\,\Gamma^1_{22}, \quad \Gamma^2_{12} = \Gamma^3_{13} = \frac{1}{2W}\frac{\partial W}{\partial r}, \\ \Gamma^2_{33} = -\sin\theta\cos\theta, \quad \Gamma^3_{23} = \cot\theta. \end{gathered} \tag{12.10}$$

From the definition $R^l_k = g^{lp}R_{pk}$, with

$$R_{pk} = \partial_n\Gamma^n_{pk} - \partial_k\Gamma^n_{pn} + \Gamma^n_{pk}\Gamma^m_{nm} - \Gamma^n_{pm}\Gamma^m_{nk}, \tag{12.11}$$

combined with (12.10), we find that

$$R_0^0 = \frac{1}{2UV}\frac{d^2U}{dr^2} - \frac{1}{4UV^2}\frac{dU}{dr}\frac{dV}{dr} - \frac{1}{4U^2V}\left(\frac{dU}{dr}\right)^2 + \frac{1}{2UVW}\frac{dU}{dr}\frac{dW}{dr}, \quad (12.12)$$

$$R_1^0 = -\frac{A}{UV}\left[\frac{1}{2UW}\frac{dW}{dr}\frac{dU}{dr} - \frac{1}{W}\frac{d^2W}{dr^2} + \frac{1}{2W^2}\left(\frac{dW}{dr}\right)^2 + \frac{1}{2VW}\frac{dV}{dr}\frac{dW}{dr}\right], \quad (12.13)$$

$$R_1^1 = \frac{1}{2UV}\frac{d^2U}{dr^2} + \frac{1}{WV}\frac{d^2W}{dr^2} - \frac{1}{4VU^2}\left(\frac{dU}{dr}\right)^2 - \frac{1}{2VW^2}\left(\frac{dW}{dr}\right)^2$$
$$- \frac{1}{4UV^2}\frac{dV}{dr}\frac{dU}{dr} - \frac{1}{2V^2W}\frac{dV}{dr}\frac{dW}{dr}, \quad (12.14)$$

$$R_2^2 = R_3^3 = \frac{1}{2VW}\frac{d^2W}{dr^2} - \frac{1}{4V^2W}\frac{dW}{dr}\frac{dV}{dr} - \frac{1}{W} + \frac{1}{4VUW}\frac{dW}{dr}\frac{dU}{dr}. \quad (12.15)$$

All the other components of R_k^i are zero. Equations (12.12)-(12.14) readily imply

$$R_1^0 = \frac{A}{U}[R_1^1 - R_0^0]. \quad (12.16)$$

Using (12.12), (12.14), and (12.15), we arrive at the following expression for R_k^k:

$$R_k^k \equiv R = \frac{1}{UV}\frac{d^2U}{dr^2} - \frac{1}{2UV^2}\frac{dV}{dr}\frac{dU}{dr} - \frac{1}{2U^2V}\left(\frac{dU}{dr}\right)^2 + \frac{2}{VW}\frac{d^2W}{dr^2}$$
$$+ \frac{1}{UVW}\frac{dU}{dr}\frac{dW}{dr} - \frac{1}{2VW^2}\left(\frac{dW}{dr}\right)^2 - \frac{1}{V^2W}\frac{dV}{dr}\frac{dW}{dr} - \frac{2}{W}. \quad (12.17)$$

Now let us assume that the source of gravitational field is described by the energy-momentum tensor of a perfect fluid

$$T^{mn} = \sqrt{-g}\,[(\rho + p)\,u^m u^n - g^{mn}p)], \quad (12.18)$$

where ρ is the fluid's density, p the isotropic pressure, and u^m the 4-vector of velocity of the fluid. Since the object is spherically symmetric and static, the quantities ρ and p depend only on r, $u^\alpha = 0$ ($\alpha = 1, 2, 3$), for u^0 and u_0 we have (in view of the identity $g_{mn}u^m u^n = 1$), respectively,

$$u^0 = \frac{1}{\sqrt{g_{00}}} = \frac{1}{\sqrt{U}}, \quad u_0 = \sqrt{U}. \quad (12.19)$$

From (12.18) and (12.19) it follows that the nonzero components of tensor T_n^m are

$$T_0^0 = \sqrt{-g}\,\rho, \quad T_1^0 = -\sqrt{-g}\,(\rho + p)\,\frac{A(r)}{U},$$
$$T_1^1 = T_2^2 = T_3^3 = -\sqrt{-g}\,p. \quad (12.20)$$

For the problem at hand the system of Hilbert-Einstein equations assumes the form

$$\left(R_0^0 - \frac{1}{2}R\right) = 8\pi\rho, \quad (12.21)$$

$$R_1^0 = -8\pi(\rho + p)\,A(r)/U, \quad (12.22)$$

$$\left(R_1^1 - \frac{1}{2}R\right) = -8\pi p, \quad (12.23)$$

$$\left(R_2^2 - \frac{1}{2}R\right) = -8\pi p. \quad (12.24)$$

Since for R_1^0 we have (12.16), Eq. (12.22) implies

$$\frac{A(r)}{U}\left[(R_1^1-R_0^0)+8\pi(\rho+p)\right]=0. \tag{12.25}$$

This equation is always valid in view of (12.21) and (12.23).

We have, therefore, arrived at an important result, namely, that for a static and spherically symmetric problem the metric coefficient $g_{01}(r)=-A(r)$ is not determined by the Hilbert-Einstein equations and, hence, in the framework of GR this coefficient may be an arbitrary function of r. This arbitrariness leads to a situation in which the Riemann space-time is not well-defined.

Let us now consider the remaining equations (12.21), (12.23), and (12.24). Since

$$R_0^0-\frac{1}{2}R=\frac{1}{2}(R_0^0-R_1^1)-R_2^2,$$

after we substitute (12.12), (12.14), and (12.15) into Eq. (12.21) we get

$$-\frac{1}{W\left(\frac{d\sqrt{W}}{dr}\right)}\frac{d}{dr}\left[\left(\frac{d\sqrt{W}}{dr}\right)^2\frac{\sqrt{W}}{V}\right]+\frac{1}{W}=8\pi\rho.$$

This yields

$$V=\frac{\sqrt{W(r)}}{\sqrt{W(r)}-2M(W)}\left(\frac{d\sqrt{W}}{dr}\right)^2, \tag{12.26}$$

where

$$M(W)=4\pi\int\limits_{\sqrt{W(0)}}^{\sqrt{W(r)}}\rho W\,d\sqrt{W}. \tag{12.27}$$

Similarly, since

$$R_1^1-\frac{1}{2}R=-\frac{1}{2}(R_0^0-R_1^1)-R_2^2,$$

combining (12.23) with (12.12), (12.14), and (12.15), we get

$$\frac{\sqrt{W}}{V}\frac{d\sqrt{W}}{dr}\frac{d}{dr}\ln(U\sqrt{W})-1=8\pi pW. \tag{12.28}$$

Taking into account (12.26), we can write Eq. (12.28) as follows:

$$\frac{d}{d\sqrt{W}}\ln[U\sqrt{W}]=\frac{1+8\pi pW}{\sqrt{W}-2M(W)}. \tag{12.29}$$

This yields the following expression for U:

$$U=\frac{C_0}{\sqrt{W}}\exp\left[-\int\limits_{\sqrt{W(r)}}^{\sqrt{W_0}}\frac{1+8\pi pW}{\sqrt{W}-2M(W)}\,d\sqrt{W}\right]. \tag{12.30}$$

Here $W_0=W(r_0)$, with r_0 the radius of the object, and C_0 is the integration constant. Formula (12.30) can also be written in another (equivalent) form. To this end we employ the relationship $dM=4\pi\rho W d\sqrt{W}$, which follows from (12.27),

and write the integral in (12.30) in the form

$$-\int_{\sqrt{W(r)}}^{\sqrt{W_0}} \frac{1+8\pi pW}{\sqrt{W}-2M(W)}\, d\sqrt{W} = -\int_{\sqrt{W(r)}}^{\sqrt{W_0}} \frac{d(\sqrt{W}-2M(W))}{\sqrt{W}-2M(W)} - 8\pi \int_{\sqrt{W(r)}}^{\sqrt{W_0}} \frac{(\rho+p)\,W d\sqrt{W}}{\sqrt{W}-2M(W)}$$

$$= \ln \frac{\sqrt{W(r)}-2M(W)}{\sqrt{W_0}-2m} - 8\pi \int_{\sqrt{W(r)}}^{\sqrt{W_0}} \frac{(\rho+p)\,W d\sqrt{W}}{\sqrt{W}-2M(W)}.$$

Hence,

$$U = \varphi(W)\, \frac{\sqrt{W}-2M(W)}{\sqrt{W}}, \tag{12.31}$$

where

$$\varphi(W) = C \exp\Big[-8\pi \int_{\sqrt{W(r)}}^{\sqrt{W_0}} \frac{(\rho+p)\,W d\sqrt{W}}{\sqrt{W}-2M(W)}\Big], \tag{12.32}$$

and $C = C_0/(\sqrt{W_0} - 2m)$. Here and in what follows we denote by m the total mass of the object:

$$m = 4\pi \int_{\sqrt{W(0)}}^{\sqrt{W_0}} \rho W d\sqrt{W}. \tag{12.27$'$}$$

Now let us consider Eq. (12.24). Since

$$R_2^2 - \frac{1}{2} R = -\frac{1}{2}(R_0^0 + R_1^1),$$

we can write the equation as follows:

$$R_0^0 + R_1^1 = 16\pi p. \tag{12.33}$$

Substituting (12.12) and (12.14) and allowing for (12.26) and (12.30), we find that

$$-W \frac{dp}{d\sqrt{W}} = (\rho+p)\,[M(W) + 4\pi(\sqrt{W})^3 p]\,\{1 - 2M(W)/\sqrt{W}\}^{-1}. \tag{12.34}$$

The latter constitutes the basic equation of Newtonian hydrostatics and allows for gravitational corrections.

Thus, out of the three functions U, V, and W the system of equations (12.21), (12.23), and (12.24) does not determine the function $W(r)$. We have demonstrated that the Hilbert-Einstein equations for a static spherically symmetric problem are valid for arbitrary functions $A(r)$ and $W(r)$. The metric coefficients (12.6) satisfying Eqs. (12.21)-(12.24) have the form

$$g_{00} = \varphi(W)\, \frac{\sqrt{W}-2M(W)}{\sqrt{W}}, \quad g_{01} = -A(r),$$

$$g_{11} = -\frac{\sqrt{W}}{\sqrt{W}-2M(W)} \Big[\Big(\frac{d\sqrt{W}}{dr}\Big)^2 - \frac{A^2}{\varphi(W)}\Big], \tag{12.35}$$

$$g_{22} = -W(r), \quad g_{33} = -W(r)\sin^2\theta,$$

with $\varphi(W)$ given by (12.32). Since g_{22} must tend to γ_{22} as $r \to \infty$, (12.3) and (12.6) imply that for large r's

$$W \simeq r^2. \tag{12.36}$$

Suppose that the matter considered here fills the volume of a sphere of radius r_0. Since by W_0 we denote the value of $W(r)$ at point $r = r_0$, for $W > W_0$ we have

$$p = \rho = 0. \tag{12.37}$$

We now bring in the following condition:

$$g_{00}(\sqrt{W} \to \infty) \to \gamma_{00} = 1. \tag{12.38}$$

Combining (12.32) and (12.35) with (12.37), we see that (12.38) gives the following value of the constant C in (12.32): $C = 1$. Hence, a function $\varphi(W)$ guaranteeing that condition (12.38) is met must have the form

$$\varphi(W) = \exp\Big[-8\pi \int_{\sqrt{W(r)}}^{\sqrt{W_0}} \frac{(\rho+p)\,W d\sqrt{W}}{\sqrt{W}-2M(W)}\Big]. \tag{12.39}$$

The set of metric coefficients defined in (12.35) shows that to each pair of functions $W(r)$ and $A(r)$ depending on the distance r in the Minkowski space-time there corresponds a certain gravitational field and, hence, a certain effective Riemann space-time. Thus, the *Hilbert-Einstein equations alone do not enable us to determine the gravitational field and, hence, the effective Riemann space-time unambiguously*. Let us illustrate this assertion by an example. For the sake of simplicity we will consider the region outside the object. In this case $\varphi(W) = 1$, $M(W) = m = \text{const}$, and, hence,

$$U = \frac{\sqrt{W}-2m}{\sqrt{W}}, \quad V = \frac{\sqrt{W}}{\sqrt{W}-2m}\left(\frac{d\sqrt{W}}{dr}\right)^2, \tag{12.40}$$

and the metric coefficients specified in (12.35) are

$$g_{00} = \frac{\sqrt{W}-2m}{\sqrt{W}}, \quad g_{01} = -A(r), \quad g_{11} = -\frac{\sqrt{W}}{\sqrt{W}-2m}\Big[\left(\frac{d\sqrt{W}}{dr}\right)^2 - A^2\Big],$$
$$g_{22} = -W(r), \quad g_{33} = -W(r)\sin^2\theta. \tag{12.41}$$

Let us take two sets of functions $W(r)$ and $A(r)$:

$$W(r) = r^2, \quad A(r) = 0, \tag{12.42}$$

and

$$W(r) = (r + m + \lambda)^2, \quad A(r) = 0. \tag{12.43}$$

In (12.43) $\lambda \neq -m$ is an arbitrary parameter, which means that by assigning it different values we obtain a one-parameter family of functions. Knowing the functions $W(r)$ and $A(r)$ (given by (12.42) and (12.43)), we can use (12.40) to find two sets of solutions:

$$U(r) = (1 - 2m/r), \quad V(r) = (1 - 2m/r)^{-1}, \tag{12.44}$$

and

$$U(r) = \frac{r+\lambda-m}{r+\lambda+m}, \quad V(r) = \frac{r+\lambda+m}{r+\lambda-m}. \tag{12.45}$$

Expressions (12.42) and (12.44), as well as expressions (12.43) and (12.45), depend on the same radial variable r and have the same asymptotic behavior as

$r \to \infty$, but constitute *two different solutions* to the Hilbert-Einstein equations. These solutions generate different line elements (12.5) and, hence, different effective Riemann space-times. For solution (12.42), (12.44) the line element (12.5) is

$$ds_1^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2\theta \, d\varphi^2), \qquad (12.46)$$

and for solution (12.43), (12.45), this quantity is

$$ds_2^2 = \frac{r+\lambda-m}{r+\lambda+m} dt^2 - \frac{r+\lambda+m}{r+\lambda-m} dr^2 - (r+m+\lambda)^2 (d\theta^2 + \sin^2\theta \, d\varphi^2). \quad (12.47)$$

The choice of solutions is not a trivial operation, since the nature of physical processes depends on this choice. Below we will directly verify this statement.

As is well known, Riemannian geometry is determined by fixing a symmetric metric tensor g_{ik} on a manifold with arbitrary but fully definite arithmetization. Transition to any other arithmetization of the manifold is carried out via a one-to-one transformation of coordinates and a respective transformation of quantities according to the tensor law. Of course, a description of Riemannian geometry is possible in principle in any admissible coordinate system of maps, but for every system taken the components of the metric tensor in the system are fully determined. Naturally, the possibility of describing the geometry of space-time in any admissible system of maps does not mean that this description is of equal simplicity in any system. Depending on the structure of Riemannian geometry, there exist systems of maps in which the description is especially simple.

How does Riemannian geometry come into play in GR? For a given arithmetization of space-time, that is, for a given system of maps, the Hilbert-Einstein equations must be solved. However, there are only ten Hilbert-Einstein equations, while there are 14 variables. This means that four components of the metric tensor (two in the case of spherical symmetry) remain arbitrary. Thus, in a given system of maps the Hilbert-Einstein equations are satisfied for any four arbitrarily chosen functions. But this means that the metric is not completely defined on the manifold, so that the Riemannian geometry contains an arbitrariness, which clearly cannot be eliminated by common coordinate transformations since no matter what coordinates are chosen the ambiguity cannot disappear. In GR this ambiguity is eliminated by arbitrary selection of the four unknown components of the metric tensor. It is not usually realized that the choice predetermines the physical content of the problem, since there is no way in which such a choice can be made generally covariant in GR. The selection in GR of the way in which the four unknown metric coefficients are chosen is known as the selection of coordinate conditions. But the coordinate conditions commonly used in GR are completely arbitrary, and their choice literally depends on the "tastes" of the researcher. All this prompts the conclusion that GR in principle cannot give unambiguous predictions of gravitational effects, which constitutes one of its main disadvantages.

The situation in RTG is quite different, since the system of coordinates in space-time is fixed by the metric tensor γ_{ik} and the gravitational field satisfies the general-covariant equation (8.37). In view of this the arbitrariness in determining the effective metric tensor g_{ik} is not present and the physical consequences are unambiguous.

The main characteristic of the Riemann space-time is the Riemann-Christoffel curvature tensor

$$R_{kmpq} = g_{nk} (\partial_q \Gamma^n_{mp} - \partial_p \Gamma^n_{mq} + \Gamma^l_{mp} \Gamma^n_{lq} - \Gamma^l_{mq} \Gamma^n_{lp}). \qquad (12.48)$$

Substituting (12.40), we find the nonzero components of the curvature tensor outside matter:

$$\begin{aligned} &R_{0101} = -\frac{m}{2W^{5/2}}\left(\frac{dW}{dr}\right)^2, \quad R_{0202} = m\frac{\sqrt{W}-2m}{\sqrt{W}},\\ &R_{0212} = -\frac{mA(r)}{\sqrt{W}}, \quad R_{0303} = R_{0202}\sin^2\theta,\\ &R_{0313} = R_{0212}\sin^2\theta, \quad R_{1212} = \frac{m}{\sqrt{W}-2m}\left[A^2(r) - \frac{1}{4W}\left(\frac{dW}{dr}\right)^2\right],\\ &R_{1313} = R_{1212}\sin^2\theta, \quad R_{2323} = 2m\sqrt{W}\sin^2\theta. \end{aligned} \tag{12.49}$$

We see that on the solutions (12.40) to the Hilbert-Einstein equations the components of R_{kmpq} are expressed in terms of the arbitrary functions $A(r)$ and $W(r)$.

Let us now calculate the invariants of the curvature tensor. Generally, on the solutions to equations $R_{mn} = 0$ the nonzero invariants have the form (Petrov, 1966, and Weinberg, 1972):

$$\begin{aligned} &I_1 = R^{lmnk}R_{lmnk}, \quad I_2 = \frac{1}{\sqrt{-g}}\,\varepsilon^{lk}{}_{mn}R^{mnpq}R_{lkpq},\\ &I_3 = R_{lmnk}R^{nkpq}R_{pq}{}^{lm}, \quad I_4 = \frac{1}{\sqrt{-g}}\,R_{lmnk}R^{nkpq}\varepsilon_{pq}{}^{ij}R_{ij}{}^{lm}, \end{aligned} \tag{12.50}$$

where ε^{lmnk} is the totally antisymmetric unit tensor.

Substituting (12.49) yields

$$I_1 = 12\left(\frac{2m}{W^{3/2}}\right)^2, \quad I_2 = I_4 = 0, \quad I_3 = 12\left(\frac{2m}{W^{3/2}}\right)^3. \tag{12.51}$$

A remark concerning (12.35) is in order. Suppose that

$$A(r) = B(W)\frac{d\sqrt{W}}{dr},$$

where $B(W)$ is an arbitrary function of $W(r)$, with $W(r)$ satisfying condition (12.36) when r is large (in all other respects $W(r)$ is an arbitrary function of r). Then the nonzero metric coefficients specified in (12.35) are

$$\begin{gathered} g_{00} = \varphi(W)\frac{\sqrt{W}-2M(W)}{\sqrt{W}}, \quad g_{01} = -B(W)\frac{d\sqrt{W}}{dr},\\ g_{11} = -\frac{\sqrt{W}}{\sqrt{W}-2M(W)}\left[1 - \frac{B^2(W)}{\varphi(W)}\right]\left(\frac{d\sqrt{W}}{dr}\right)^2,\\ g_{22} = -W, \quad g_{33} = -W\sin^2\theta, \end{gathered}$$

with the result that (12.5) can be written as follows:

$$\begin{aligned} ds^2 = {}&\varphi(W)\frac{\sqrt{W}-2M(W)}{\sqrt{W}}\,dt^2 - 2B(W)\,d\sqrt{W}\,dt\\ &-\frac{\sqrt{W}}{\sqrt{W}-2M(W)}\left(1 - \frac{B^2(W)}{\varphi(W)}\right)\left(d\sqrt{W}\right)^2 - W(d\theta^2 + \sin^2\theta\,d\varphi^2). \end{aligned}$$

We see that the dependence of ds^2 on r has vanished an $\sqrt{W}$ acquires the meaning of spatial distance in the effective Riemann space-time.

This analysis of the Hilbert-Einstein equations as applied to the given problem shows that their solutions contain two arbitrary functions, with the result that the characteristics (12.41), (12.49), and (12.51) of the effective Riemann space-time are not well-defined. In Chapter 18 we will see that a solution containing arbitrary functions yields different physical results depending on the choice of these functions, which means that GR in principle is not able to provide specific predictions concerning gravitational effects. At the same time the predictions of RTG are physically well-defined and unambiguous: *RTG contains the Hilbert-Einstein equations* (8.36) *and the general covariant-field equations* (8.37), the latter determining the structure of the gravitational field. What is even more important is that *in RTG all field variables in Eqs.* (8.36), (8.37) *depend on the spatial-temporal coordinates in the Minkowski universe.*

This last fact is a consequence of our fundamental hypothesis that the gravitational field is a physical field, is characterized by a certain value of the energy-momentum density, and like all other physical fields must be described in the Minkowski space-time. The general-covariant equations (8.37) completely eliminate the ambiguity in the solutions, whereby the effective Riemann space-time, which appears in RTG in view of the geometrization principle, is determined uniquely.

Now let us analyze Eqs. (8.37). To write these equations for functions U, V, W, and A explicitly, we find the metric tensor density $\tilde{g}^{mn}$. Formulas (12.7) and (12.8) yield

$$\begin{gathered}\tilde{g}^{00}=\frac{W}{\sqrt{UV}}\left(V-\frac{A^2}{U}\right)\sin\theta,\quad \tilde{g}^{01}=-\frac{AW}{\sqrt{UV}}\sin\theta,\\ \tilde{g}^{11}=-\sqrt{UV^{-1}}\,W\sin\theta,\quad \tilde{g}^{22}=-\sqrt{UV}\sin\theta,\quad \tilde{g}^{33}=-\sqrt{UV}/\sin\theta.\end{gathered}\tag{12.52}$$

To determine $A(r)$ it has proved convenient to consider Eqs. (8.37) in Galilean coordinates of an inertial reference frame:

$$\partial_m\tilde{g}^{mn}=0.\tag{12.53}$$

For $n=0$ this yields

$$\partial_\alpha\tilde{g}^{\alpha 0}=0,\tag{12.54}$$

where we have allowed for the fact that the components of g^{mn} are time independent. Employing the tensor transformation law, we can establish that the components of $g^{\alpha 0}$ in Galilean coordinates can be expressed in terms of the components in the spherical coordinates (12.7) as follows:

$$g^{0\alpha}=-\frac{A(r)}{UV}\frac{X^\alpha}{r},\quad \sqrt{-g}=\frac{1}{r^2}\sqrt{UV}\,W,\tag{12.55}$$

where X^α ($\alpha=1, 2, 3$) are the spatial Cartesian coordinates. From (12.55) we obtain

$$\tilde{g}^{0\alpha}=-\frac{AW}{\sqrt{UV}}\frac{X^\alpha}{r^3}.\tag{12.56}$$

Integrating (12.54) over the spherical volume and allowing for (12.56), we find, in view of the divergence theorem, that

$$4\pi\frac{A(r)W(r)}{\sqrt{UV}}=0.\tag{12.57}$$

Since Eq. (12.54) is valid both inside and outside matter, condition (12.57) should be met for any value of r. Since $W(r)$ is nonzero, (12.57) implies

$$A(r) = 0. \tag{12.58}$$

It is convenient to write the other corollaries that follow from (8.37) in spherical coordinates. To this end we write (8.37) in the following form:

$$D_m \tilde{g}^{mn} = \partial_m \tilde{g}^{mn} + \gamma^n_{mp} \tilde{g}^{mp} = 0.$$

Allowing for (12.4), (12.52), and (12.58), we find that

$$\frac{d}{dr}\left(\sqrt{UV^{-1}}\, W'\right) = 2r\sqrt{UV}. \tag{12.59}$$

Combining this with (12.26) and (12.31) yields

$$\frac{d}{dr}\left[\sqrt{\varphi}\sqrt{W}\left(\sqrt{W} - 2M\right)\frac{dr}{d\sqrt{W}}\right] = 2r\sqrt{\varphi}\,\frac{d\sqrt{W}}{dr}. \tag{12.60}$$

Below the solution to this equation is sought as $r = r(\sqrt{W})$, that is, r as a function of $\sqrt{W}$, whereby it is convenient to write Eq. (12.60) as

$$\frac{d}{d\sqrt{W}}\left[\sqrt{\varphi}\sqrt{W}\left(\sqrt{W} - 2M\right)\frac{dr}{d\sqrt{W}}\right] = 2r\sqrt{\varphi}. \tag{12.61}$$

If we introduce the notation $\Phi = r\sqrt{\varphi}$, we can write (12.61) as

$$\frac{d}{d\sqrt{W}}\left[\sqrt{W}\left(\sqrt{W} - 2M\right)\frac{d\Phi}{d\sqrt{W}} - \sqrt{W}\left(\sqrt{W} - 2M\right)\Phi\,\frac{d\ln\sqrt{\varphi}}{d\sqrt{W}}\right] = 2\Phi. \tag{12.62}$$

Since (12.39) implies

$$\frac{d\ln\sqrt{\varphi}}{d\sqrt{W}} = 4\pi\,\frac{(\rho + p)\,W}{\sqrt{W} - 2M},$$

from (12.62) we find that

$$\frac{d}{d\sqrt{W}}\left[\sqrt{W}\left(\sqrt{W} - 2M\right)\frac{d\Phi}{d\sqrt{W}} - 4\pi W^{3/2}(\rho + p)\,\Phi\right] = 2\Phi. \tag{12.63}$$

A general requirement of the solution $r = r(\sqrt{W})$ to Eq. (12.63) is that this solution must be continuous and monotonic. Suppose that the mass of the object is concentrated inside a ball of radius r_0. Let us consider the solution to Eq. (12.63) outside the ball, $r_0 \leqslant r \leqslant \infty$. The corresponding range for $\sqrt{W}$ will be $\sqrt{W_0} \leqslant \sqrt{W} < \infty$. Since within this range $p = \rho = 0$, $M(W) = m$, and $\varphi(W) = 1$, we have $\Phi(W) = r(W)$, and Eq. (12.63) assumes the form

$$\frac{d}{d\sqrt{W}}\left[\sqrt{W}\left(\sqrt{W} - 2m\right)\frac{dr}{d\sqrt{W}}\right] = 2r. \tag{12.64}$$

The general solution to this equation is the sum of two particular solutions (Belinfante, 1955, and Belinfante and Garrison, 1962)

$$1 + \frac{\sqrt{W} - m}{2m}\ln\frac{\sqrt{W} - 2m}{\sqrt{W}}, \tag{12.65}$$

$$\frac{1}{m}\left(\sqrt{W} - m\right) \tag{12.66}$$

and has the form

$$r\left(\sqrt{W}\right) = C_1\left[1 + \frac{\sqrt{W} - m}{2m}\ln\frac{\sqrt{W} - 2m}{\sqrt{W}}\right] + \frac{C_2}{m}\left(\sqrt{W} - m\right), \tag{12.67}$$

where C_1 and C_2 are arbitrary constant numbers. Since (12.36) must be valid when r becomes large, we find that $C_2 = m$. To determine the value of C_1, we must find the solution to Eq. (12.63) inside matter. Let us write this equation in expanded form:

$$\sqrt{W}\,(\sqrt{W}-2M)\,\frac{d^2\Phi}{(d\sqrt{W})^2}+2\left[\sqrt{W}-M-2\pi\,(\sqrt{W})^3\,(3\rho+p)\right]\frac{d\Phi}{d\sqrt{W}}$$
$$-2\left[1+6\pi W\,(\rho+p)+2\pi\,(\sqrt{W})^3\,\frac{d}{d\sqrt{W}}\,(\rho+p)\right]\Phi=0. \qquad (12.68)$$

The investigation of the solution to this equation inside matter is a very important problem in itself, but it depends largely on the type of functions $\rho = \rho\,(\sqrt{W})$ and $p = p\,(\sqrt{W})$, whereby we will not consider it here.

In what follows we select a solution that remains finite for all finite values of $\sqrt{W}$. Then we must put $C_1 = 0$ in (12.67), which yields

$$r\,(\sqrt{W})=\sqrt{W}-m. \qquad (12.69)$$

Combining (12.69) with (12.40) yields

$$U\,(r)=\frac{r-m}{r+m}, \quad V\,(r)=\frac{r+m}{r-m}. \qquad (12.70)$$

Thus, the RTG system of equations (8.36), (8.37) provides an unambiguous means for determining all the metric coefficients:

$$U\,(r)=\frac{r-m}{r+m}, \quad V\,(r)=\frac{r+m}{r-m},$$
$$W\,(r)=(r+m)^2, \quad A\,(r)=0. \qquad (12.71)$$

Then, in view of (12.6) and (12.71), formula (12.5) yields

$$ds^2=\frac{r-m}{r+m}\,dt^2-\frac{r+m}{r-m}\,dr^2-(r+m)^2\,(d\theta^2+\sin^2\theta\,d\varphi^2). \qquad (12.72)$$

The correspondence principle implies that the mass defined by (12.27′), m, must be equal to the active gravitational mass of the object. Formula (12.72) clearly shows that ds^2 is singular at $r = m$ and loses all physical meaning for $r < m$. This means that in RTG there can be no static spherically symmetric objects with a radius equal to or less than m. This fact provides for a lower bound on $\sqrt{W}$, namely, $\sqrt{W} > 2m$. The components (12.49) of the curvature tensor and the invariants (12.51) for solution (12.71) are uniquely defined and are

$$R_{0101}=\frac{2m}{(r+m)^3}, \quad R_{0202}=m\,\frac{r-m}{(r+m)^2}, \quad R_{0303}=R_{0202}\sin^2\theta, \qquad (12.73)$$

$$R_{1212}=-\frac{m}{r-m}, \quad R_{1313}=R_{1212}\sin^2\theta, \quad R_{2323}=2m\,(r+m)\sin^2\theta;$$

$$I_1=12\left[\frac{2m}{(r+m)^3}\right]^2, \quad I_3=12\left[\frac{2m}{(r+m)^3}\right]^3. \qquad (12.74)$$

Note that the well-known GR solution for a static spherically symmetric object, found by K. Schwarzschild, namely,

$$g_{00}\,(r)=U\,(r)=1-\frac{2m}{r},$$
$$g_{11}\,(r)=-V\,(r)=-\left(1-\frac{2m}{r}\right)^{-1}, \qquad (12.75)$$
$$g_{22}\,(r)=-W\,(r)=-r^2,$$

is not a solution to the RTG system of equations given our choice of the metric tensor γ^{ik}. Indeed, it can easily be shown that the functions (12.75) do not satisfy Eq. (12.64) which was obtained from the field equations (8.37). Therefore, Schwarzschild's line element

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2\theta \, d\varphi^2) \qquad (12.76)$$

cannot serve as the line element in the RTG effective Riemann space-time. Note that according to Schwarzschild's solution (12.75), in GR there can be no static spherically symmetric objects with a radius equal to or less than $2m$.

12.2 The Exterior Axisymmetric Solution for a Spinning Mass*

Let us now consider the solution to the RTG equations for a spinning mass m with angular momentum ma. This problem was solved within the GR framework by Kerr, 1963, who found the following expression for the line element outside the mass (the vacuum solution):

$$\begin{aligned} ds^2 = {} & \left(1 - \frac{2mB}{\rho^2}\right) d\tau^2 - 2dB\, d\tau - \rho^2 d\theta^2 \\ & - \frac{1}{\rho^2} [(B^2 + a^2)^2 - \Delta a^2 \sin^2\theta] \sin^2\theta \, d\phi^2 \\ & + 2a \sin^2\theta \, dB\, d\phi + \frac{4maB}{\rho^2} \sin^2\theta \, d\tau \, d\phi, \end{aligned} \qquad (12.77)$$

where $\rho^2 = B^2 + a^2 \cos^2\theta$, $\Delta = B^2 - 2mB + a^2$, and $(\tau, B, \theta, \phi) = \{\xi^i_{(K)}\}$ are the Kerr coordinates.

Kerr's solution (12.77) does not satisfy the complete system of RTG equations (8.36), (8.37). This can easily be verified if for the coordinates in the Minkowski space-time we take $\xi^i_{(K)}$, as required by RTG, and substitute the metric coefficients from (12.77) into the system of equations (8.37).

To obtain an exterior axisymmetric solution to the complete system of RTG equations for a spinning mass, we employ an approach that enables us, first, to use the already found solution (12.77) to the system of Hilbert-Einstein equations for constructing a solution that would satisfy system (8.36), (8.37) and, second, to unambiguously determine the range of variables $\xi^i_{(K)}$ in the effective Riemann space-time. The latter fact is especially important for understanding many physical phenomena and, as will be demonstrated in Chapter 13 using the gravitational collapse as an example, marks a significant difference between the predictions of RTG and those of GR.

Note that the approach suggested here for finding the solution of the complete RTG system of equations can be applied in all cases when the solutions to the Hilbert-Einstein equations are known in terms of some system of coordinates y^i. The system of equations (8.37) establishes a one-to-one relationship between the coordinates y^i and the Minkowski space-time coordinates x^i. This means that there can be only such effective Riemann space-times as are specified on a single map, say, in Cartesian coordinates of the Minkowski space-time.

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* See Vlasov and Logunov, 1987.

In what follows it is expedient to use the somewhat modified Kerr coordinates

$$\{\xi^i\} = \{\tau, B, \theta, \varphi\}, \tag{12.78}$$

$$\varphi = \phi - \tan^{-1}\left(\frac{B-m}{a}\right) + \frac{\pi}{2}. \tag{12.79}$$

In terms of these new coordinates ξ^i (below we will call them the Kerr coordinates), the nonzero components of $g_{ik}(\xi^i)$ are

$$\begin{aligned}
&g_{00}(\xi^i) = 1 - \frac{2mB}{\rho^2}, \quad g_{01}(\xi^i) = -\frac{1}{\rho^2(\Delta+m^2)}[\rho^2 m^2 + (\rho^2 - 2mB)(B^2+a^2)],\\
&g_{03}(\xi^i) = \frac{2maB}{\rho^2}\sin^2\theta, \quad g_{22}(\xi^i) = -\rho^2,\\
&g_{11}(\xi^i) = -\frac{a^2\sin^2\theta}{\rho^2(\Delta+m^2)^2}[2\rho^2 m^2 + (\rho^2 - 2mB)(B^2+a^2)],\\
&g_{13}(\xi^i) = \frac{a\sin^2\theta}{\rho^2(\Delta+m^2)}[\rho^2 m^2 - 2mB(B^2+a^2)],\\
&g_{33}(\xi^i) = -\frac{\sin^2\theta}{\rho^2}[\rho^2(B^2+a^2) + 2ma^2 B\sin^2\theta],
\end{aligned} \tag{12.80}$$

and the nonzero components of $g^{ik}(\xi^i)$ are

$$\begin{aligned}
&g^{00}(\xi^i) = -\frac{a^2\sin^2\theta}{\rho^2}, \quad g^{01}(\xi^i) = -\frac{B^2+a^2}{\rho^2},\\
&g^{03}(\xi^i) = \frac{ma(2B-m)}{\rho^2(\Delta+m^2)}, \quad g^{11}(\xi^i) = -\frac{\Delta}{\rho^2},\\
&g^{13}(\xi^i) = -\frac{m^2 a}{\rho^2(\Delta+m^2)}, \quad g^{22}(\xi^i) = -\frac{1}{\rho^2},\\
&g^{33}(\xi^i) = -\frac{1}{\rho^2\sin^2\theta} + \frac{2a^2}{\rho^2(\Delta+m^2)} - \frac{\Delta a^2}{\rho^2(\Delta+m^2)^2}.
\end{aligned} \tag{12.81}$$

Let us denote the Cartesian (Galilean) coordinates in the Minkowski space-time by x^i and assume that the Kerr coordinates ξ^i are functions of x^i. The natural requirements that functions of transformations $\xi^i \leftrightarrow x^i$ in RTG must satisfy are the continuity and one-to-one nature of these functions in the entire Minkowski universe.

The concrete form of the relationship between the ξ^i and the coordinates x^i of the Minkowski space-time can be found from Eqs. (8.37), which establish both the physical meaning and the range of the ξ^i (initially the solutions to the Hilbert-Einstein equations were given in terms of the Kerr coordinates ξ^i).

System (8.37) can be represented identically in Cartesian coordinates in the following manner:

$$\frac{1}{\sqrt{-g(x)}}\frac{\partial}{\partial x^m}[\sqrt{-g(x)}\,g^{mn}(x)] \equiv -g^{pq}(x)\,\Gamma^n_{pq}(x) = 0, \tag{12.82}$$

where $\Gamma^n_{pq}(x)$ is defined by (12.9). Applying the transformation law to $\Gamma^n_{pq}(x)g^{pq}(x)$ for the substitution of $\xi^i(x)$ for x^i, we obtain from (12.82) the following:

$$\Gamma^n_{pq}(x)g^{pq}(x) = -\Box_\xi\, x^n = 0, \quad n = 0, 1, 2, 3, \tag{12.83}$$

where $\Box_\xi$ is the covariant d'Alembertian operator in the Kerr variables ξ^i:

$$\Box_\xi = \frac{1}{\sqrt{-g(\xi)}} \frac{\partial}{\partial \xi^p}\left[\sqrt{-g(\xi)}\, g^{pq}(\xi) \frac{\partial}{\partial \xi^q}\right]$$
$$= -\frac{1}{\rho^2}\left\{a^2 \sin^2\theta \partial_0^2 + 2(B^2+a^2)\partial_0\partial_1 \right.$$
$$+ \frac{2ma(m-2B)}{\Delta+m^2}\partial_0\partial_3 + 2B\partial_0 + \Delta\partial_1^2 + 2(B-m)\partial_1$$
$$- \frac{2m^2a(B-m)}{(\Delta+m^2)^2}\partial_3 + \frac{2m^2a}{\Delta+m^2}\partial_1\partial_3$$
$$\left. + \left[\frac{\Delta a^2}{(\Delta+m^2)^2} - \frac{2a^2}{\Delta+m^2} + \frac{1}{\sin^2\theta}\right]\partial_3^2 + \partial_2^2 + \cot\theta\,\partial_2\right\}.$$

Here $\partial_0 = \frac{\partial}{\partial\tau}$, $\partial_1 = \frac{\partial}{\partial B}$, $\partial_2 = \frac{\partial}{\partial\theta}$, and $\partial_3 = \frac{\partial}{\partial\varphi}$.

We will seek the variable $x^0 \equiv t$ in the form

$$t = \tau + f(B). \tag{12.84}$$

Substitution into (12.83) yields the following equation for $f(B)$:

$$\Delta \frac{d^2 f(B)}{dB^2} + 2(B-m)\frac{df(B)}{dB} + 2B = 0, \tag{12.85}$$

which can easily be integrated. The solution is

$$f(B) = -\left\{B + \frac{m}{\sqrt{m^2-a^2}}\left[B_+ \ln\frac{B-B_+}{B_+} - B_- \ln\frac{B-B_-}{B_-}\right]\right\}, \tag{12.86}$$

where we have introduced the notation

$$B_\pm = m \pm \sqrt{m^2-a^2}. \tag{12.87}$$

We see that a real solution for $t = \tau + f(B)$ exists only if

$$B > B_+, \tag{12.88}$$

with $B \to B_+$ as $t \to \infty$. Thus, the Kerr variable $\xi^1 = B$ admits only the following values: $B_+ < B \leqslant \infty$. It is readily noticed that

$$x^3 = (B-m)\cos\theta \tag{12.89}$$

is a solution to Eq. (12.83). What remains to be found is x^1 and x^2. We will seek them in the form $x^1 = Z(B)\cos\varphi\sin\theta$ and $x^2 = Z(B)\sin\varphi\sin\theta$. Equation (12.83) for x^1 then becomes

$$\cos\varphi\left[\Delta\frac{d^2Z}{dB^2} + 2(B-m)\frac{dZ}{dB} - \left(\frac{a^2\Delta}{(\Delta+m^2)^2} - \frac{2a^2}{\Delta+m^2} + 2\right)Z\right]$$
$$- \sin\varphi\frac{2m^2a}{\Delta+m^2}\left[\frac{dZ}{dB} - \frac{B-m}{\Delta+m^2}Z\right] = 0, \tag{12.90}$$

while for x^2 the equation is the same except that $\sin\varphi$ is substituted for $\cos\varphi$ and $\cos\varphi$ for $-\sin\varphi$. Since the two equations must be valid for any value of φ, we necessarily arrive at

$$\frac{dZ}{dB} - \frac{B-m}{\Delta+m^2}Z = 0 \tag{12.91}$$

and

$$\Delta \frac{d^2Z}{dB^2} + 2(B-m)\frac{dZ}{dB} - \left(\frac{a^2\Delta}{(\Delta+m^2)^2} - \frac{2a^2}{\Delta+m^2} + 2\right) Z = 0. \qquad (12.92)$$

Solving Eq. (12.91), we get

$$Z(B) = \sqrt{(B-m)^2 + a^2}, \qquad (12.93)$$

which also satisfies Eq. (12.92).

We have, therefore, found the following solutions to Eq. (12.83):

$$\begin{aligned} x^0 &\equiv t = \tau - \left\{B + \frac{m}{\sqrt{m^2-a^2}}\left[B_+ \ln\frac{B-B_+}{B_+} - B_- \ln\frac{B-B_-}{B_-}\right]\right\}, \\ x^1 &= \sqrt{(B-m)^2+a^2}\cos\varphi\sin\theta,\ x^2 = \sqrt{(B-m)^2+a^2}\sin\varphi\sin\theta, \\ x^3 &= (B-m)\cos\theta. \end{aligned} \qquad (12.94)$$

This also establishes the relationship between the Kerr variables $\{\xi^i\}$ and the Galilean (Cartesian) coordinates x^i in the Minkowski space-time.

Introducing the spheroidal coordinates $\{\tilde{x}^i\} = (t,\ r = B - m,\ \theta,\ \varphi)$ in (12.94) instead of the Galilean coordinates and allowing for the metric

$$\gamma_{mn} = \left(1,\ -\frac{r^2+a^2\cos^2\theta}{r^2+a^2},\ -(r^2+a^2\cos^2\theta),\ -(r^2+a^2)\sin^2\theta\right), \quad (12.95)$$

we obtain the following relationships linking the Kerr variables $(\tau, B, \theta, \varphi)$ with the spheroidal variables (t, r, θ, φ):

$$\begin{gathered} \tau = t + \left\{(r+m) + \frac{m}{\sqrt{m^2-a^2}}\left[(r_+ + m)\ln\frac{r-r_+}{r_+ + m} - (r_- + m)\ln\frac{r-r_-}{r_- + m}\right]\right\}, \\ B = r + m, \end{gathered} \qquad (12.96)$$

where $r_\pm = \pm\sqrt{m^2-a^2}$. According to (12.88), the range of r is

$$r_+ < r \leqslant \infty, \qquad (12.97)$$

and the Kerr variables θ and φ assume the values $0 \leqslant \theta \leqslant \pi$ and $0 \leqslant \varphi < 2\pi$, since they are simply the spherical angular variables θ and φ in the Minkowski space-time.

From (12.96) it follows that the coorrespondence between the Kerr variables $\{\xi^i\} = \{\tau, B, \theta, \varphi\}$ and the spheroidal coordinates $\{\tilde{x}^i\} = \{t, r, \theta, \varphi\}$ is one-to-one and the Jacobian of the respective transformation, $\partial(\xi)/\partial(\tilde{x})$, is equal to unity.

After finding the functions $\xi^i(\tilde{x})$, which are solutions to the system of equations (8.37), it is easy to establish, on the basis of (12.80) and (12.81), the explicit form of the solution to the complete system of RTG equations for the metric coefficients of the effective Riemann space-time outside the spinning mass. To this end it is sufficient to employ the tensor transformation law

$$g^{ik}(\tilde{x}) = \frac{\partial\tilde{x}^i}{\partial\xi^m}\frac{\partial\tilde{x}^k}{\partial\xi^n}g^{mn}(\xi). \qquad (12.98)$$

Allowing here for (12.96) and (12.81), we obtain

$$g^{ik}(\widetilde{x}) = \begin{pmatrix} \frac{(B^2+a^2)^2 - a^2\Delta\sin^2\theta}{\Delta\rho^2} & 0 & 0 & \frac{2maB}{\Delta\rho^2} \\ 0 & -\frac{\Delta}{\rho^2} & 0 & -\frac{am^2}{\rho^2(\Delta+m^2)} \\ 0 & 0 & -\frac{1}{\rho^2} & 0 \\ \frac{2maB}{\Delta\rho^2} & -\frac{am^2}{\rho^2(\Delta+m^2)} & 0 & -\frac{1}{\rho^2\sin^2\theta}+\frac{2a^2}{\rho^2(\Delta+m^2)}-\frac{\Delta a^2}{\rho^2(\Delta+m^2)^2} \end{pmatrix} \tag{12.99}$$

In a similar manner it can be demonstrated that the nonzero components of tensor $g_{ik}(\widetilde{x})$ are

$$\begin{aligned} &g_{00}(\widetilde{x}) = 1 - \frac{2mB}{\rho^2}, \quad g_{01}(\widetilde{x}) = -\frac{2a^2m^3B\sin^2\theta}{\rho^2\Delta(\Delta+m^2)}, \\ &g_{03}(\widetilde{x}) = \frac{2maB\sin^2\theta}{\rho^2}, \quad g_{22}(\widetilde{x}) = -\rho^2, \\ &g_{11}(\widetilde{x}) = -\frac{\rho^2}{\Delta} - \frac{a^2m^4\sin^2\theta}{\Delta^2\rho^2(\Delta+m^2)^2}[\rho^2\Delta + 2mB(B^2+a^2)], \\ &g_{13}(\widetilde{x}) = \frac{m^2a\sin^2\theta}{\Delta\rho^2(\Delta+m^2)}[\rho^2(B^2+a^2)+2mBa^2\sin^2\theta], \\ &g_{33}(\widetilde{x}) = -\frac{\sin^2\theta}{\rho^2}[\rho^2(B^2+a^2)+2mBa^2\sin^2\theta]. \end{aligned} \tag{12.100}$$

In (12.99) and (12.100) the quantities B, ρ^2, and Δ are assumed to be known functions of variable r. Note that $g^{ik}(\widetilde{x})$ and $g_{ik}(x)$ do not depend on time t.

At $a = 0$ the solution (12.100) is transformed into solution (12.71) for the metric coefficients of the effective Riemann space-time outside a spherically symmetric object.

The exterior axisymmetric solution for a spinning electrically charged mass was found within the RTG framework by Karabut and Chugreev, 1987. Below we give the main results of the study, which considered a spinning mass m with angular momentum ma, charge Q, and magnetic moment $\mu = |Q|a$. Naturally, the problem requires solving simultaneously the RTG equations (8.36), (8.37) and the Maxwell equations for the electromagnetic field

$$\nabla_k F^{ik} = -4\pi j^i, \tag{12.101}$$

$$\nabla_l F_{ik} + \nabla_i F_{kl} + \nabla_k F_{li} = 0, \tag{12.102}$$

where as usual ∇_k is the symbol for the covariant derivative with respect to the metric g_{mn} of the effective Riemann space-time. The only nonzero energy-momentum tensor of matter outside the object is the electromagnetic-field energy-momentum tensor

$$T_i^k = -\frac{1}{4\pi}F_{il}F^{kl} + \frac{1}{16\pi}\delta_i^k F_{lm}F^{lm}. \tag{12.103}$$

Within the GR framework the exterior solution to this problem has been found by Newman and Kerr, 1965. In terms of the Kerr coordinates this solu-

tion is

$$ds^2 = \left(1 - \frac{2mB - Q^2}{\rho^2}\right) d\tau^2 - 2\, dB\, d\tau$$
$$- \rho^2 (d\theta)^2 - \frac{1}{\rho^2} [(B^2 + a^2)^2 - \tilde{\Delta} a^2 \sin^2\theta] \sin^2\theta\, (d\phi)^2 \qquad (12.104)$$
$$+ 2a \sin^2\theta\, dB\, d\phi + \frac{2a}{\rho^2} (2mB - Q^2) \sin^2\theta\, d\tau\, d\phi,$$

where $\tilde{\Delta} = B^2 - 2mB + a^2 + Q^2$, and the other quantities coincide with those introduced in (12.77).

The electromagnetic field-strength tensor can be written in terms of the Kerr coordinates as

$$F \equiv F_{ik}\, dx^i \wedge dx^k$$
$$= -\frac{2Q}{\rho^4} \{(B^2 - a^2\cos^2\theta)\, dB \wedge d\tau - 2a^2 B \cos\theta \sin\theta\ d\theta \wedge d\tau$$
$$- a\sin^2\theta\, (B^2 - a^2\cos^2\theta)\, dB \wedge d\phi$$
$$+ 2aB\, (B^2 + a^2) \sin\theta \cos\theta\, d\theta \wedge d\phi\}. \qquad (12.105)$$

The Newman-Kerr solution (12.104) does not satisfy the system of equations (8.37) in the Kerr coordinates $\{\xi^i_{(K)}\}$ and is, therefore, not a solution to the RTG equations.

Proceeding in the same manner as was done above in deriving the relationship linking $\{x^i\}$ and $\{\xi^i\}$, Karabut and Chugreev, 1987, derived, on the basis of Eqs. (8.37), the following formulas:

$$x^0 \equiv t = \tau - \left\{B + \frac{m}{\sqrt{m^2 - a^2 - Q^2}} \left[\tilde{B}_+ \ln \frac{B - \tilde{B}_+}{\tilde{B}_+} - \tilde{B}_- \ln \frac{B - \tilde{B}_-}{\tilde{B}_-}\right]\right.$$
$$\left. - \frac{Q^2}{2\sqrt{m^2 - a^2 - Q^2}} \ln \frac{B - \tilde{B}_+}{B - \tilde{B}_-}\right\}, \qquad (12.106)$$
$$x^1 = [(B - m)^2 + a^2]^{1/2} \cos\varphi \sin\theta,$$
$$x^2 = [(B - m)^2 + a^2]^{1/2} \sin\varphi \sin\theta,$$
$$x^3 = (B - m)\cos\theta,$$

where

$$\tilde{B}_\pm = m \pm \sqrt{m^2 - a^2 - Q^2}$$

and

$$\varphi = \phi - \tan^{-1}\left(\frac{B - m}{a}\right) + \frac{\pi}{2}.$$

We see that t is real only if $B > \tilde{B}_+$, with $t \to \infty$ as $B \to \tilde{B}_+$.

If we introduce, as before, the spheroidal coordinates $\{\tilde{x}^i\} = \{t, r = B - m, \theta, \varphi\}$ in the Minkowski space-time, we arrive at the following relationships between $\{\tau, B, \theta, \varphi\}$ and $\{t, r, \theta, \varphi\}$:

$$\tau = t + \left\{(r + m) + \frac{m}{\sqrt{m^2 - a^2 - Q^2}} \left[(\tilde{r}_+ + m) \ln \frac{r - \tilde{r}_+}{\tilde{r}_+ + m} - (\tilde{r}_- + m) \ln \frac{r - \tilde{r}_-}{\tilde{r}_- + m}\right]\right.$$
$$\left. - \frac{Q^2}{2\sqrt{m^2 - a^2 - Q^2}} \ln \frac{r - \tilde{r}_+}{r - \tilde{r}_-}\right\}, \qquad (12.107)$$
$$B = r + m,$$

where $\tilde{r}_\pm = \pm\sqrt{m^2 - a^2 - Q^2}$.

The range of values of r, according to the inequality $B > \widetilde{B}_+$, is bounded below:

$$r > \widetilde{r}_+. \tag{12.108}$$

Now it is easy to find the explicit form of the solution to the RTG system of equations outside a spinning charged object for the metric coefficients of the effective Riemann space-time. To this end one must only apply the tensor law of transformations of the Newman-Kerr metric coefficients, bearing in mind the coordinate transformations (12.107).

The nonzero components of $g_{ik}(\widetilde{x})$, which are solutions to the complete system of RTG equations, are (Karabut and Chugreev, 1987)

$$\begin{aligned}
&g_{00}(\widetilde{x}) = 1 - \frac{2mB - Q^2}{\rho^2}, \quad g_{22}(\widetilde{x}) = -\rho^2, \quad g_{33}(\widetilde{x}) = -\lambda\frac{\sin^2\theta}{\rho^2},\\
&g_{01}(\widetilde{x}) = \frac{a^2(Q^2-m^2)(2mB-Q^2)\sin^2\theta}{\rho^2\widetilde{\Delta}(\widetilde{\Delta}+m^2-Q^2)}, \quad g_{03}(\widetilde{x}) = \frac{a(2mB-Q^2)\sin^2\theta}{\rho^2},\\
&g_{11}(\widetilde{x}) = -\frac{\rho^2}{\widetilde{\Delta}} - \lambda\frac{a^2(Q^2-m^2)^2\sin^2\theta}{\rho^2\widetilde{\Delta}^2(\widetilde{\Delta}+m^2-Q^2)},\\
&g_{13}(\widetilde{x}) = -\lambda\frac{a(Q^2-m^2)\sin^2\theta}{\rho^2\widetilde{\Delta}(\widetilde{\Delta}+m^2-Q^2)},
\end{aligned} \tag{12.109}$$

with $\lambda = (B^2 + a^2)^2 - a^2\widetilde{\Delta}\sin^2\theta$. Here B, ρ^2, and $\widetilde{\Delta}$ are assumed to be known functions of r.

Obviously, $g_{ik}(\widetilde{x})$ is independent of t. Assuming that $a = 0$ in (12.109), we arrive at the exterior solution for an electrically charged spherically symmetric object. The expression for the line element in the effective Riemann space-time in spherical coordinates of the Minkowski space-time is

$$\begin{aligned}
ds^2 = &\left(\frac{r-m}{r+m} + \frac{Q^2}{(r+m)^2}\right)dt^2 - \left(\frac{r-m}{r+m} + \frac{Q^2}{(r+m)^2}\right)^{-1}dr^2\\
&-(r+m)^2(d\theta^2 + \sin^2\theta\, d\varphi^2),
\end{aligned} \tag{12.110}$$

with r, θ, and φ varying within the following ranges:

$$\sqrt{m^2-Q^2} < r \leqslant \infty, \quad 0 \leqslant \theta \leqslant \pi, \quad 0 \leqslant \varphi < 2\pi.$$

Applying the tensor transformation law to (12.105), we can find the electromagnetic field-strength tensor in spheroidal coordinates $\{\widetilde{x}^i\}$:

$$\begin{aligned}
F = &-\frac{2Q}{\rho^4}(B^2 - a^2\cos^2\theta)\,dr \wedge (dt - a\sin^2\theta\, d\varphi)\\
&+\frac{4Q}{\rho^4}aB\cos\theta\sin\theta\, d\theta \wedge \left(a dt - (B^2+a^2)\,d\varphi - \frac{a(B^2+a^2)(Q^2-m^2)}{\widetilde{\Delta}(\widetilde{\Delta}+m^2-Q^2)}dr\right).
\end{aligned} \tag{12.111}$$

Since the components of the electric field vector E_μ and the magnetic induction vector B^μ are defined as

$$E_\mu = F_{0\mu} \text{ and } B^\mu = -\frac{1}{2}\left[\det\left(\frac{g_{0\alpha}g_{0\beta}}{g_{00}} - g_{\alpha\beta}\right)\right]^{-1/2} e^{\mu\nu\sigma}F_{\nu\sigma},$$

we can easily establish, on the basis of (12.109) and (12.111), the asymptotic behavior of these components for large r's. In the dipole approximation we

have

$$\begin{aligned} &E_r \simeq \frac{Q}{r^2}+O\left(\frac{1}{r^4}\right), && B^r \simeq \frac{2aQ\cos\theta}{r^4}+O\left(\frac{1}{r^5}\right), \\ &E_\theta \simeq O\left(\frac{1}{r^4}\right), && B^\theta \simeq \frac{aQ\sin\theta}{r^4}+O\left(\frac{1}{r^6}\right), \\ &E_\varphi = O, && B^\varphi \simeq O\left(\frac{1}{r^7}\right). \end{aligned} \tag{12.112}$$

Note that such a field is created by an object with charge Q and magnetic moment $\mu = |Q|a$.

Chapter 13. Gravitational Collapse

Within the framework of general relativity (Landau and Lifshitz, 1975, Weinberg, 1972, and Zel'dovich and Novikov, 1971, 1974) the conclusion is drawn that if a massive star has burned out its nuclear fuel but has not lost a sufficient fraction of its mass, there are no forces that can stop it from contracting under gravitation, with the result that the density of the star will tend to infinity over a finite interval of proper (or local) time. This stage in the evolution of certain stars has become known as gravitational collapse. Misner, Thorne, and Wheeler, 1973 (Box 18.1), consider gravitational collapse and the emerging singularity as "the greatest single crisis of physics, central to understanding the nature of matter and the universe."

In this chapter, following Vlasov and Logunov, 1985a, 1986a, we will show how RTG changes the entire nature of gravitational collapse and leads to the phenomenon of gravitational time dilation, due to which the contraction of a massive object takes a finite proper time lapse in the comoving reference frame and, most important, the density of matter remains constant and does not exceed 10^{16} g/cm^3, the luminosity of the object reduces, or the object "blackens", but nothing unusual happens to the object. Thus, the predictions of RTG differ drastically from those of GR. Below we give a brief description of the results of gravitational collapse that follow from GR.

The line element in the reference frame comoving with a nonstatic spherically symmetric object can be represented as follows:

$$ds^2 = d\tau^2 - e^{\omega(R,\tau)}\,dR^2 - B^2(\tau, R)\,(d\theta^2+\sin^2\theta\,d\varphi^2), \tag{13.1}$$

where τ is the proper time, and R, θ and φ are the spherical coordinates in the comoving reference frame. It is expedient for our discussion to introduce the universal notation

$$\xi^i = (\tau, R, \theta, \varphi). \tag{13.2}$$

According to (13.1), the nonzero metric coefficients $g_{mn}(\xi)$ are

$$\begin{aligned} &g_{00}(\xi)=1,\ g_{11}(\xi)=-e^{\omega(\tau,R)},\ g_{22}(\xi)=-B^2(\tau, R), \\ &g_{33}(\xi)=-B^2(\tau, R)\sin^2\theta. \end{aligned} \tag{13.3}$$

Using (13.3), we can easily find the $g^{mn}(\xi)$ coefficients that are nonzero:

$$\begin{aligned} &g^{00}(\xi)=1,\ g^{11}(\xi)=-e^{-\omega(\tau,R)},\ g^{22}(\xi)=-B^{-2}(\tau, R), \\ &g^{33}(\xi)=-B^{-2}(\tau, R)\sin^{-2}\theta. \end{aligned} \tag{13.4}$$

The functions $\omega(\tau, R)$ and $B(\tau, R)$ can be found from the Hilbert-Einstein equations.

Following Oppenheimer and Snyder, 1939, let us consider the simplest variant of gravitational collapse of spherically symmetric dustlike matter with zero pressure. The energy-momentum tensor density in this case is

$$T^{mn} = \sqrt{-g}\,\rho(\tau, R)\,u^m u^n, \tag{13.5}$$

with $\rho(\tau, R)$ the self-energy density, and u^m the 4-vector of velocity.

Oppenheimer and Snyder, 1939, demonstrated that if ρ is independent of R, then in the comoving reference frame, or the frame in which

$$u^0 = \frac{d\tau}{ds} = 1,\ u^1 = \frac{dR}{d\tau} = 0,\ u^2 = \frac{d\theta}{d\tau} = 0,\ u^3 = \frac{d\varphi}{d\tau} = 0, \tag{13.6}$$

the simplest exact solution to the Hilbert-Einstein system of equations has the form of the Tolman solution:

$$B = R\left(1 - \frac{\tau}{\tau_0}\right)^{2/3} \text{ if } R \leqslant R_0, \tag{13.7}$$

$$B = \left(R^{3/2} - R_0^{3/2}\frac{\tau}{\tau_0}\right)^{2/3} \text{ if } R \geqslant R_0, \tag{13.8}$$

$$e^{\omega(\tau, R)} = \left(\frac{\partial B}{\partial R}\right)^2, \tag{13.9}$$

where

$$R_0^3 = \frac{9}{2} m\tau_0^2, \tag{13.10}$$

with m the active gravitational mass of the object. Solutions (13.7) and (13.8) imply that the range of values of τ is bounded above by the value $\tau = \tau_0$ *and that $B(\tau, R)$ may assume all values from 0 to* ∞.

For the density of matter we have the following formula:

$$\rho(\tau) = \frac{1}{6\pi(\tau - \tau_0)^2}, \tag{13.11}$$

which shows that $\rho(\tau)$ becomes infinite at $\tau = \tau_0$.

Radial fall of test bodies in metric (13.3), (13.7)-(13.9) obeys the following equations (Oppenheimer and Snyder, 1939):

$$\frac{dB}{d\tau} = -\sqrt{\frac{2m}{B}},\quad \frac{d^2B}{d\tau^2} = -\frac{m}{B^2}.$$

These equations show that the collapse of the falling dust particles to the Schwarzschild radius $B_g = 2m$ occurs over a finite proper time interval and, meeting nothing in their movement through empty space, the particles reach the center $B = 0$ simultaneously (which results in the energy density ρ becoming infinite). Within the framework of GR the conclusion is drawn that a nonstatic object may have dimensions smaller than $2m$. From the viewpoint of an external observer the region inside the Schwarzschild sphere is "cut off" from the observer, any object that finds itself inside the sphere of radius $2m$ is gravitationally "short-circuited", and no light can escape from the inner region of the sphere. Such objects, which have infinite densities but possess no material boundaries, became known in GR as "black holes".

We can now summarize. To an external observer a spherically symmetric object with a sufficiently large mass $m > 3M_\odot$, where $M_\odot$ is the Sun's mass, will appear

as contracting without limit, approaching the dimensions of the Schwarzschild sphere of radius $B_g = 2m$ over of an infinite time interval (in the reference frame linked with the external observer!). From the standpoint of the observer comoving with collapsing matter the situation is quite qifferent. Here the surface of the sphere of radius $2m$ is not material and the "falling" observer crosses it and reaches the sphere's center in the course of a finite time interval; at the center the energy density becomes infinite.

From the viewpoint of GR, everything that happens to matter inside the Schwarzschild sphere cannot in principle be cognized by the external observer. Physically this situation is intolerable since it imposes limits on the knowledge of how matter evolves.

Let us now turn to the picture drawn by RTG. The underlying geometry for the gravitational field is the geometry of the Minkowski space-time. The components of the gravitational field or, in view of (8.1), the components of the metric tensor g^{mn} obey the universal field equations (8.37), in addition to obeying the Hilbert-Einstein equations. Therefore, only a solution that satisfies both the system of equations (8.36) and the system of equations (8.37) has physical meaning because it is the system of equations (8.37) that takes into account the fundamental role of the Minkowski space-time, separates inertia from gravitation, and fixes the structure of the gravitational field as a Faraday-Maxwell physical field possessing spins 2 and 0.

It can easily be verified that solutions (13.7)-(13.9) do not satisfy the system of equations (8.37) if in spherical coordinates the metric tensor of the Minkowski space-time is given in the form (12.3). To study any problem in the RTG framework, one must solve Eqs. (8.36), (8.37) in terms of the coordinates of the Minkowski space-time.

A transfer from one set of coordinates to another in the Minkowski space-time is carried out by a one-to-one transformation with a nonzero Jacobian. RTG contains none of the complicated topologies inherent in theories dealing with the Riemann space-time.

Let us consider a spherically symmetric line element of the general form

$$ds^2 = g_{00}(t, r)\,dt^2 + 2g_{01}(t, r)\,dt\,dr + g_{11}(t, r)\,dr^2 - \overline{B}^2(t, r)(d\theta^2 + \sin^2\theta\,d\varphi^2), \tag{13.12}$$

where t is the temporal variable, and r, θ, and φ are the spherical coordinates of the Minkowski space-time, which set of variables we denote by x^i:

$$x^i = (t, r, \theta, \varphi).$$

We now shift from variables x^i to variables ξ^i, assuming that

$$\tau = \tau(t, r) \text{ and } R = R(t, r). \tag{13.13}$$

Since the metric coefficients $g_{mn}(x^i)$ in (13.12) are linked with the metric coefficients (13.3) by a tensor transformation law, we obtain

$$\begin{aligned}
g_{00}(t, r) &= \left(\frac{\partial\tau}{\partial t}\right)^2 - e^{\omega(\tau, R)}\left(\frac{\partial R}{\partial t}\right)^2,\\
g_{01}(t, r) &= \frac{\partial\tau}{\partial t}\frac{\partial\tau}{\partial r} - e^{\omega(\tau, R)}\frac{\partial R}{\partial t}\frac{\partial R}{\partial r},\\
g_{11}(t, r) &= \left(\frac{\partial\tau}{\partial r}\right)^2 - e^{\omega(\tau, R)}\left(\frac{\partial R}{\partial r}\right)^2,\\
\overline{B}(t, r) &= B(\tau, R).
\end{aligned} \tag{13.14}$$

For all admissible transformation functions (13.13) these expressions by definition automatically satisfy the Hilbert-Einstein equations if solutions (13.7)-(13.9) are taken into account. But in RTG the metric coefficients (13.14) must, in addition, satisfy the general-covariant system of equations (8.37), whose solution makes it possible to find the explicit form of functions (13.13) and, hence, to obtain a solution to the complete system of equations (8.36), (8.37) in terms of coordinates of the Minkowski space-time.

Following Vlasov and Logunov, 1985a, 1985b, below we give a more detailed analysis of all these aspects than is given in Chapter 12. The presentation will demonstrate the limiting nature of the coordinate conditions used in GR and how RTG lifts this limitation.

We write the system of equations (8.37) in a somewhat different form. To this end we turn to the well-known equality

$$\Gamma^q_{kl}(x)\ g^{kl}(x) = -\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^p}[\sqrt{-g}\,g^{pq}(x)], \tag{13.15}$$

with $\Gamma^q_{kl}(x)$ defined in (12.9). If we employ the law of transformation for $\Gamma^q_{kl}(x)g^{kl}(x)$, the change in variables (13.13) yields

$$\Gamma^q_{kl}(x)\,g^{kl}(x) = -\,\Box\, x^q, \tag{13.16}$$

where $\Box$ is the generalized d'Alembertian operator,

$$\Box = \frac{1}{\sqrt{-g(\xi)}}\frac{\partial}{\partial \xi^p}\left[\sqrt{-g(\xi)}\,g^{pq}(\xi)\,\frac{\partial}{\partial \xi^q}\right]. \tag{13.17}$$

Comparing (13.15) and (13.16), we find that

$$\Box\, x^q = \frac{1}{\sqrt{-g(x)}}\frac{\partial}{\partial x^p}[\sqrt{-g(x)}\,g^{pq}(x)],\ \ q=0,\ 1,\ 2,\ 3. \tag{13.18}$$

De Donder, 1921, 1926, and later Fock, 1939, 1957, in solving island problems, employed noncovariant harmonic conditions of the type

$$\frac{\partial}{\partial x^p}[\sqrt{-g(x)}\,g^{pq}(x)] = 0 \tag{13.19}$$

as preferred coordinate conditions. But why must these conditions be written in the Riemann space-time in terms of Cartesian coordinates? Neither de Donder nor Fock could provide an explanation, since there cannot in principle be any global Cartesian coordinates in a Riemannian geometry, and they had no idea of the fundamental importance of the Minkowski space-time to gravity.

On the basis of (13.18) we can write condition (13.19) as follows:

$$\Box x^q = 0,\quad q = 0,\ 1,\ 2,\ 3. \tag{13.20}$$

For this reason the coordinates x^q satisfying (13.20) became known as "harmonic".

Remaining within the framework of GR, condition (13.20) cannot be made covariant. In harmonic coordinates the Hilbert-Einstein system of equations simplifies considerably, which apparently prompted Fock to call this system preferred. If we wish to retain condition (13.19) and make it universal, we must write it in covariant form, which as noted earlier is impossible in GR. The new field equations can be found if we turn to the physical structure of the gravitational field. In Chapter 8 this approach led us to Eq. (8.3),

$$\frac{1}{\sqrt{-g(x)}}D_p[\sqrt{-g(x)}\,g^{pq}(x)] = \frac{1}{\sqrt{-g(x)}}\frac{\partial}{\partial x^p}[\sqrt{-g(x)}\,g^{pq}(x)] + \gamma^q_{mn}(x)\,g^{mn}(x) = 0, \tag{13.21}$$

which thanks to (13.18) can be written thus:

$$\Box x^q = -\gamma^q_{mn}(x)\frac{\partial x^m}{\partial \xi^p}\frac{\partial x^n}{\partial \xi^l}g^{pl}(\xi), \quad q = 0, 1, 2, 3, \tag{13.22}$$

where

$$\gamma^q_{mn}(x) = \frac{1}{2}\gamma^{qp}(\partial_m\gamma_{pn} + \partial_n\gamma_{pm} - \partial_p\gamma_{mn}) \tag{13.23}$$

is the Christoffel symbol for the Minkowski space-time. In the case of Galilean coordinates, $\gamma^q_{mn} = 0$ and Eqs. (13.22) coincide with Eqs. (13.20).

Thus, our equations, (13.22), imply that the harmonic coordinates defined in (13.20) coincide with the Galilean (Cartesian) coordinates of the Minkowski space-time. This assertion reflects the fundamental role and field origin of the effective Riemann space-time. Equations (13.22) are covariant, reflect the true structure of the gravitational field, and stress the fundamental nature of the Minkowski space-time. The role of these equations is highly important since they change the nature of the predicted phenomena. New physics emerges as a result, especially in the case of strong fields. This will become especially evident when we study gravitational collapse of massive objects and in Chapter 16, where we study the time evolution of a homogeneous and isotropic universe.

But let us return to Eqs. (13.22), which constitute an alternative form for the general-covariant RTG field equations (8.37). Employing formulas (12.3) for the metric coefficients γ^{mn}, formulas (12.4) for the connection coefficients γ^q_{mn}, and formulas (13.4) for the $g^{pq}(\xi)$ and substituting all this into (13.22), we get

$$\frac{\partial}{\partial\tau}\left(e^{\omega/2}B^2\frac{\partial t}{\partial\tau}\right) = \frac{\partial}{\partial R}\left(e^{-\omega/2}B^2\frac{\partial t}{\partial R}\right), \tag{13.24}$$

$$\frac{\partial}{\partial\tau}\left(e^{\omega/2}B^2\frac{\partial r}{\partial\tau}\right) = \frac{\partial}{\partial R}\left(e^{-\omega/2}B^2\frac{\partial r}{\partial R}\right) - 2re^{\omega/2}. \tag{13.25}$$

If we employ solutions (13.7)-(13.9), these equations assume the form

$$\frac{\partial}{\partial\tau}\left(\frac{\partial B}{\partial R}B^2\frac{\partial t}{\partial\tau}\right) = \frac{\partial}{\partial R}\left[\left(\frac{\partial B}{\partial R}\right)^{-1}B^2\frac{\partial t}{\partial R}\right], \tag{13.26}$$

$$\frac{\partial}{\partial\tau}\left(\frac{\partial B}{\partial R}B^2\frac{\partial r}{\partial\tau}\right) = \frac{\partial}{\partial R}\left[\left(\frac{\partial B}{\partial R}\right)^{-1}B^2\frac{\partial r}{\partial R}\right] - 2r\frac{\partial B}{\partial R}. \tag{13.27}$$

By solving these equations we can establish the relationship that exists between the coordinates R and τ of the comoving reference frame and the coordinates r and t of the Minkowski space-time. Therefore, the solutions fix the explicit form of the admissible functions (13.13).

From physical considerations it is clear that for a solution to exist the transformations (13.13) must be one-to-one and the temporal axes t and τ must point in the same direction. These requirements lead us to the following solutions (Vlasov and Logunov, 1985a) to Eqs. (13.26), (13.27):

$$t = \tau - 2\sqrt{2mB} + 2m\ln\frac{\sqrt{B}+\sqrt{2m}}{\sqrt{B}-\sqrt{2m}}, \tag{13.28}$$

$$r = B - m. \tag{13.29}$$

Here for the sake of brevity we have not substituted the explicit form of the function $B = B(\tau, R)$.

Since the ranges of the variables $x^n = (t, r, \theta, \varphi)$, which are the coordinates in the Minkowski space-time, are fixed, the ranges of $\tau(t, r)$ and $R(t, r)$ in RTG must be specified in accordance with the ranges of the Minkowski space-time variables x^n.

The solution (13.28) as applied to the exterior problem directly implies that matter in a nonstatic spherically symmetric RTG problem occupies, at any time t, a sphere of radius B that is always greater than $2m$. The value $B = 2m$ is attained as t is sent to infinity. Thus, we can always write

$$B(\tau, R) > 2m. \tag{13.30}$$

Generally, the physical meaning of the variables ξ^n, in terms of which the Hilbert-Einstein equations are written, is established when we link them to the coordinates x^n of the Minkowski space-time via (13.22); the ξ^n become an additional coordinate system in the Minkowski space-time. Here, if the solutions g_{mn} to the Hilbert-Einstein equations hold true in a certain range Ω of variables ξ^n, then Eqs. (13.22) usually narrow the range of ξ^n, and these variables, in view of (13.22), are functions not only of the coordinates x^n but of the gravitational field as well. The spatial-temporal region Ω^* in which the ξ^n may vary, defined in (13.22), does not coincide with Ω.

Thus, while in terms of x^n with the metric tensor $\gamma_{mn}(x)$ the system of equations (8.36) has a solution $g'_{mn}(x)$, in terms of ξ^n with the metric tensor

$$\gamma_{mn}(\xi) = \frac{\partial x^p}{\partial \xi^m} \frac{\partial x^k}{\partial \xi^n} \gamma'_{pk}(x) \tag{13.31}$$

the same system of equations has a solution $g_{mn}(\xi)$ only in Ω^*. Hence, not every solution $g_{mn}(\xi)$ to the Hilbert-Einstein equations with $\xi^n \in \Omega$ satisfies the system of equations (13.22) or, which is the same, the system of equations (8.37).

Let us now establish the form of the metric tensor γ_{mn} for the Minkowski space-time in terms of the coordinates $\xi^n = (\tau, R, \theta, \varphi)$ of the comoving reference frame by employing the tensor transformation law (13.31). Here, according to (12.3), $\gamma'_{pk}(x) = (1, -1, -r^2, -r^2\sin^2\theta)$ and the relationship between (t, r) and (τ, R) is fixed by (13.28) and (13.29).

We calculate the transformation matrix

$$\frac{\partial x^p}{\partial \xi^n} = \begin{pmatrix} \dfrac{\partial t}{\partial \tau} & \dfrac{\partial r}{\partial \tau} & 0 & 0 \\ \dfrac{\partial t}{\partial R} & \dfrac{\partial r}{\partial R} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{13.32}$$

Combining (13.8) with (13.28) and (13.29), we obtain

$$\begin{aligned} \frac{\partial t}{\partial \tau} &= \frac{B(\tau, R)}{B(\tau, R) - 2m}, \quad \frac{\partial r}{\partial \tau} = -\sqrt{\frac{2m}{B(\tau, R)}}, \\ \frac{\partial t}{\partial R} &= -\frac{\sqrt{2mR}}{B(\tau, R) - 2m}, \quad \frac{\partial r}{\partial R} = \sqrt{\frac{R}{B(\tau, R)}}. \end{aligned} \tag{13.33}$$

In view of (13.33), it can easily be verified that

$$\det\left(\frac{\partial x^p}{\partial \xi^n}\right) = \sqrt{\frac{R}{B(\tau, R)}} \neq 0. \tag{13.34}$$

Substituting (13.33) into (13.31), we arrive at the following expression for $\gamma_{mn}(\xi)$:

$$\gamma_{mn}(\xi)=\begin{pmatrix} \left(\frac{\partial t}{\partial\tau}\right)^2-\left(\frac{\partial r}{\partial\tau}\right)^2 & \frac{\partial t}{\partial\tau}\frac{\partial t}{\partial R}-\frac{\partial r}{\partial\tau}\frac{\partial r}{\partial R} & 0 & 0 \\ \frac{\partial t}{\partial\tau}\frac{\partial t}{\partial R}-\frac{\partial r}{\partial\tau}\frac{\partial r}{\partial R} & \left(\frac{\partial t}{\partial R}\right)^2-\left(\frac{\partial r}{\partial R}\right)^2 & 0 & 0 \\ 0 & 0 & -(B-m)^2 & 0 \\ 0 & 0 & 0 & -(B-m)^2\sin^2\theta \end{pmatrix}. \quad (13.35)$$

On the basis of (13.33) we can write

$$\gamma_{00}(\xi)=\frac{B(\tau,\ R)}{B(\tau,\ R)-2m}\left[1+\frac{2(2m)^2}{B^2(\tau,R)}\ \frac{B(\tau,\ R)-m}{B(\tau,\ R)-2m}\right], \quad (13.36)$$

$$\gamma_{01}(\xi)=-\frac{\sqrt{2mR}}{B(\tau,\ R)}\left[\frac{B^2(\tau,\ R)}{(B(\tau,\ R)-2m)^2}-1\right], \quad (13.37)$$

$$\gamma_{11}(\xi)=-R\ \frac{[B(\tau,R)-2m]^2-2mB(\tau,\ R)}{B(\tau,\ R)\,[B(\tau,\ R)-2m]^2}, \quad (13.38)$$

$$\det\gamma_{mn}(\xi)=-\frac{R}{B(\tau,\ R)}[B(\tau,\ R)-m]^4\sin^2\theta. \quad (13.39)$$

For $B>2m$ the metric $\gamma_{mn}(\xi)$ has no singularities. A singularity in $\gamma_{mn}(\xi)$ emerges at $B=2m$.

Employing formula (13.35) for the metric tensor $\gamma_{mn}(\xi)$, we can find the expression for the line element in the Minkowski space-time in terms of the comoving coordinates ξ^n:

$$d\sigma^2=\gamma_{mn}(\xi)d\xi^m d\xi^n. \quad (13.40)$$

We write it in the form

$$d\sigma^2=d\zeta^2-dl^2, \quad (13.41)$$

where

$$d\zeta=\sqrt{\gamma_{00}(\xi)}\,d\tau+\frac{\gamma_{01}(\xi)}{\sqrt{\gamma_{00}(\xi)}}\,dR \quad (13.42)$$

and

$$dl^2=\chi_{\alpha\beta}\,d\xi^\alpha\,d\xi^\beta, \quad (13.43)$$

with

$$\chi_{\alpha\beta}=-\gamma_{\alpha\beta}+\frac{\gamma_{0\alpha}\gamma_{0\beta}}{\gamma_{00}}. \quad (13.44)$$

Combining (13.35)-(13.38) with (13.44), we get

$$\begin{gathered}\chi_{11}=R\frac{(B-2m)^2}{(B-2m)(B^2+4m^2)+4m^2B},\quad \chi_{22}=(B-m)^2,\\ \chi_{33}=(B-m)^2\sin^2\theta,\quad \chi_{\alpha\beta}=0\quad\text{if}\quad\alpha\neq\beta.\end{gathered} \quad (13.45)$$

From (13.41) it follows that ζ is the physical time (Logunov, 1985) and, therefore, must be real while dl^2 must be positive, since it is the square of the spatial separation between two closely lying points in the ordinary three-dimensional space. In view of Sylvester's criterion, the quadratic form (13.43) will be positive definite if χ_{11}, χ_{22}, and χ_{33} are positive. This implies $B>2m$.

Allowing for (13.45) in the definition (13.43) of dl^2, we obtain

$$dl^2 = \frac{R(B-2m)^2}{(B-2m)(B^2+4m^2)+4m^2B}\,dR^2 + (B-m)^2(d\theta^2+\sin^2\theta\,d\varphi^2). \qquad (13.46)$$

Both (13.42) and (13.46) imply that a transition to a comoving reference frame in the Minkowski space-time is possible only if $B > 2m$. This means that, *in accordance with RTG (see (13.29)), in nature there cannot be nonstatic spherically symmetric objects with a radius r equal to or less than m, with the result that there can be no gravitational "short-circuiting" and matter cannot disappear from our space.* In other words, according to RTG there can be physical objects with a fairly large mass and possessing an inner structure. This conclusion greatly distinguishes RTG from GR.

Using formulas (13.8), (13.9), (13.28), and (13.29) in (13.14), we find that

$$\begin{aligned} &g_{00}(r) = \frac{r-m}{r+m}, \quad g_{01}(r) = 0, \quad g_{11}(r) = -\frac{r+m}{r-m}, \\ &g_{22}(r) = -(r+m)^2, \quad g_{33}(r,\theta) = -(r+m)^2\sin^2\theta. \end{aligned} \qquad (13.47)$$

The equations of radial motion of test particles in metric (13.47) with a zero velocity and spatial infinity can be written as follows (Vlasov and Logunov, 1985a, 1986a):

$$\frac{dr}{dt} = -\frac{r-m}{r+m}\sqrt{\frac{2m}{r+m}}, \quad \frac{d^2r}{dt^2} = -\frac{m(r-m)(r-5m)}{(r+m)^4}. \qquad (13.48)$$

These equations suggest that it takes an infinite time interval for a test particle to fall on a sphere of radius $r = m$:

$$\left(\frac{dr}{dt}\right)_{r\to m} \to 0, \quad \left(\frac{d^2r}{dt^2}\right)_{r\to m} \to 0. \qquad (13.49)$$

This effect can be called *gravitational time dilation* (Vlasov and Logunov, 1985a, 1986a).

Let us now return to formulas (13.28) and (13.29). We reintroduce the universal gravitational constant and write the expression for proper time in terms of the variables t and r of the Minkowski space-time:

$$\tau = t + 2\sqrt{2mG(r+mG)} - 2mG\ln\frac{\sqrt{r+mG}+\sqrt{2mG}}{\sqrt{r+mG}-\sqrt{2mG}}. \qquad (13.50)$$

We see that G is included in the expression for proper time, which means that the passage of proper time τ depends on the nature of the gravitational field. Let us see how the proper time interval $d\tau$ is connected with the time interval dt of the Minkowski space-time.

Since

$$\frac{dt(\tau, B)}{d\tau} = \left.\frac{\partial t}{\partial \tau}\right|_B + \left.\frac{\partial t}{\partial B}\right|_\tau \frac{dB}{d\tau},$$

where the rate of radial fall $dB/d\tau$ in the (13.3) metric with $B > 2mG$ is

$$\frac{dB}{d\tau} = -\sqrt{\frac{2mG}{B}}$$

and, in view of (13.28),

$$\left.\frac{\partial t}{\partial \tau}\right|_B = 1 \quad \text{and} \quad \left.\frac{\partial t}{\partial B}\right|_\tau = -\sqrt{\frac{2mG}{B}}\,\frac{B}{B-2mG},$$

we obtain

$$\frac{dt}{d\tau}=\frac{B}{B-2mG}\,.$$

Going back to r via (13.29) in this formula, we find that

$$d\tau=\frac{r-mG}{r+mG}\,dt. \tag{13.51}$$

This expression shows that for a freely falling object the proper time interval $d\tau$, with a fixed time interval dt of an external observer, tends to zero as r approaches the horizon (i.e. as $r\to mG$) and, hence, all physical processes in the reference frame comoving with the falling objects slow down without limit. Actually, however, since in RTG the value of r is always greater than mG, the falling of a test body on another object occurs over a finite time interval both for the comoving observer and an external observer. Time does not cease to flow in either reference frame. Such a process is in all respects similar to the falling of a test body on the surface of a star.

We now use restriction (13.30) to find the range of admissible τ's. Since $B(R,\tau)$ and τ are linked by (13.8), restriction (13.30) means that proper time τ never reaches τ_0. From the viewpoint of an external observer the surface of a spherical star, for example, of "radius" $R=R_0$, approaches the horizon (a Schwarzschild sphere of radius $B(R_0,\tau)=2m$) over an infinite time interval t, while for the observer in the comoving reference frame this process occurs over a finite proper time interval

$$\tau_c=\left[1-\left(\frac{2m}{R_0}\right)^{3/2}\right]\tau_0. \tag{13.52}$$

This formula can easily be derived from (13.8) if we allow for (13.30). Thus, the RTG equations (8.37) limit the range of τ's in the following manner: $\tau<\tau_c<\tau_0$.

Let us calculate the limiting value of density ρ. Into (13.11) we substitute the expression (13.52) for τ_c, since this expression is valid for $R=R_0$. We then have

$$\rho_{\max}=\frac{3}{32\pi m^2}\,. \tag{13.53}$$

We see that ρ does not become infinite because the new field equations (8.37) guarantee that proper time τ does not become equal to τ_0.

Note that while solutions (13.8), (13.9) to the Hilbert-Einstein equations have meaning in the entire range of values of proper time, $0\leqslant\tau\leqslant\tau_0$, according to the RTG equations (8.37) these solutions have no physical meaning in the region from τ_c to τ_0.*

From the viewpoint of an external observer, the luminosity of a collapsing object exponentially falls off (the object "blackens"), but nothing unusual happens to the object since its density always remains finite.

* Thus, the statement of Zel'dovich and Grishchuk, 1986, that every solution to the Hilbert-Einstein equations satisfies Eqs. (8.37) and that this situation changes nothing is simply erroneous. Gravitational collapse in RTG differs drastically from gravitational collapse in GR, since in the former there can be no catastrophic contraction of matter to an infinite density in either the x^n coordinates or the comoving coordinates ξ^n. Neither can there be any singularities or regions blocked from an external observer. In this sense RTG can contain no such objects known as "black holes" in GR, objects that depend only on the mass and charge of the collapsing object and that have neither material boundaries nor inner structure. The collapse of an object in RTG asymptotically tends to a state with a finite density and a finite radius, the latter always being greater than mG. Such an object always has a material boundary and an inner structure. No gravitational "self-circuiting" exists and matter does not disappear from our space. According to RTG, in nature there can be neither static nor nonstatic spherically symmetric objects with a radius less than or equal to mG.

Despite the fact that gravitational contraction of a massive object to the size of the respective Schwarzschild sphere always takes a finite proper time interval τ_c, which is always shorter than τ_0, we can never say that the object has reached this state because in RTG this is impossible in principle, since such a state constitutes a limit and is achieved only when time t in the Minkowski space-time becomes infinite.

Let us now turn to the problem of the motion of test particles in metrics (12.99) and (12.109). First, following Vlasov and Logunov, 1987, we consider the motion in metric (12.99).

To determine the trajectories of test bodies in spheroidal coordinates $\{\widetilde{x}^i\}$ we use the Hamilton-Jacobi equations

$$\frac{d\widetilde{x}^i}{ds} = -g^{ik}\partial_k S \tag{13.54}$$

and

$$g^{ik}\partial_i S \partial_k S = 1, \tag{13.55}$$

where S is the action integral of a test particle in metric (12.99). Equation (13.55) yields

$$\begin{gathered} S = -\varepsilon t + \omega\varphi + S(r) + S(\theta), \\ \left(\frac{dS(\theta)}{d\theta}\right)^2 = k - a^2\cos^2\theta - \left(a\varepsilon\sin\theta - \frac{\omega}{\sin\theta}\right)^2, \\ \frac{dS(r)}{dr} = -\frac{1}{\Delta}\left[\frac{m^2 a\omega}{\Delta + m^2} + \sqrt{\{\varepsilon[a^2 + (r+m)^2] - a\omega\}^2 - \Delta\{k + (r+m)^2]}\right], \end{gathered} \tag{13.56}$$

where ε, ω, and k are constants. The formula for $dS(r)/dr$ corresponds to the case of falling particles.

Equation (13.54) then yields

$$\begin{gathered} \frac{dt}{ds} = \frac{1}{\Delta\rho^2}\{\varepsilon[(a^2 + (r+m)^2)^2 - a^2\Delta\sin^2\theta] - 2ma\omega(r+m)\}, \\ \frac{dr}{dt} = \frac{-\Delta\sqrt{[\varepsilon(a^2 + (r+m)^2) - a\omega]^2 - \Delta[k + (r+m)^2]}}{\varepsilon[(a^2 + (r+m)^2)^2 - a^2\Delta\sin^2\theta] - 2ma\omega(r+m)}, \\ \frac{d\varphi}{dt} = \frac{2ma\varepsilon(r+m) - a^2\omega + \Delta\omega/\sin^2\theta}{\varepsilon[(a^2 + (r+m)^2)^2 - a^2\Delta\sin^2\theta] - 2ma\omega(r+m)} \\ - \frac{m^2 a}{\Delta + m^2}\,\frac{\sqrt{[\varepsilon(a^2 + (r+m)^2) - a\omega]^2 - \Delta[k + (r+m)^2]}}{\varepsilon[(a^2 + (r+m)^2)^2 - a^2\Delta\sin^2\theta] - 2ma\omega(r+m)}, \end{gathered} \tag{13.57}$$

$$\frac{d\theta}{dt} = \frac{\Delta\sqrt{k - a^2\cos^2\theta - (a\varepsilon\sin\theta - \omega/\sin\theta)^2}}{\varepsilon[(a^2 + (r+m)^2)^2 - a^2\Delta\sin^2\theta] - 2ma\omega(r+m)}.$$

We see that as the particles fall to the horizon ($r \to r_+$, $\Delta \to 0$), time t in the Minkowski space-time tends to infinity while angle φ remains finite, or $d\varphi/dt \to 0$ and $d\varphi/dR < \infty$, that is, the particles take an infinite time to fall to the horizon and do not "wind around" the horizon. Such deceleration can be understood from a qualitative analysis of the forces acting on the particles in the vicinity of r_+.

From the expressions (13.57) for the velocity components of a particle in spheroidal coordinates we have

$$\{v^\alpha\} = \left\{\sqrt{\frac{r^2 + a^2\cos^2\theta}{r^2 + a^2}}\,\frac{dr}{dt},\ \sqrt{r^2 + a^2\cos^2\theta}\,\frac{d\theta}{dt},\ \sqrt{r^2 + a^2}\sin\theta\,\frac{d\varphi}{dt}\right\}. \tag{13.58}$$

From this it readily follows that in the vicinity of the horizon

$$v^\alpha \simeq \Delta, \ \frac{dv^\alpha}{dt} \simeq \Delta,$$

that is, as $r \to r_+$, acceleration and velocity tend to zero.

Suppose that $\theta = \pi/2$. Then for the radial component of the acceleration, specifically

$$\frac{dv^r}{dt} = \frac{d}{dt}\left(\frac{dr}{dt}\frac{r}{\sqrt{r^2+a^2}}\right) - \sqrt{r^2+a^2}\left(\frac{d\varphi}{dt}\right)^2, \tag{13.59}$$

we have in the vicinity of the horizon ($r \simeq r_+$)

$$\frac{dv^r}{dt} \simeq \Delta \frac{m^2-a^2}{2m^3 r_+^2} > 0, \tag{13.60}$$

while for large separations ($r \gg r_+$) we have

$$\frac{dv^r}{dt} \simeq \frac{2m}{r^2}\left(1 - \frac{3}{2\varepsilon^2}\right), \tag{13.61}$$

that is, relativistic particles and light ($\varepsilon^2 > 3/2$) are decelerated and nonrelativistic particles ($1 \leqslant \varepsilon^2 < 3/2$) are accelerated.

As shown by Karabut and Chugreev, 1987, the motion of a charged test body in metric (12.109), when charge e is opposite in sign to charge Q, is similar to the motion of a test body in metric (12.99). For example, when the test body approaches the horizon ($r \to \tilde{r}_+$), the components of velocity v^α and acceleration dv^α/dt tend to zero. The radial component of acceleration, dv^r/dt, is positive in the vicinity of the horizon ($r \simeq \tilde{r}_+$), while for large separations ($r \gg \tilde{r}_+$) it has the form

$$\frac{dv^r}{dt} \simeq \frac{2m}{r^2}\left(1 - \frac{3}{2\varepsilon^2}\right) - \frac{eQ}{r^2}\left(\frac{\varepsilon+1}{\varepsilon^2} - 1\right). \tag{13.62}$$

For relativistic particles $\left(\varepsilon^2 > \left(\frac{1+\sqrt{5}}{2}\right)^2 > 3/2\right)$, as demonstrated by (13.61), there is both gravitational and electrostatic deceleration, while for nonrelativistic particles ($1 < \varepsilon < 3/2$) there is acceleration, or attraction. At $\varepsilon = 1$ acceleration (13.61) is the superposition of the Newtonian and Coulomb accelerations.

In Chapter 11 we arrived at RTG equations involving a massive graviton. If the line element in the effective Riemann space-time is fixed in the form

$$ds^2 = U\,dt^2 - V\,(d\sqrt{W})^2 - W\,(d\theta^2 + \sin^2\theta\, d\varphi^2) \tag{13.63}$$

and the line element in the flat space in the form

$$d\sigma^2 = dt^2 - (r')^2\,(d\sqrt{W})^2 - r^2\,(d\theta^2 + \sin^2\theta\, d\varphi^2),$$

then the system of RTG equations (11.9), (11.10) for a spherically symmetric object can be written in the form (Vlasov and Logunov, 1988a, 1988b)

$$(\sqrt{v})' = \sqrt{\omega}\left[1 + \frac{m^2}{2}(W - r^2)\right], \tag{13.64}$$

$$v\,\frac{\omega'}{\omega} = -\frac{m^2 W}{2}\left[1 - v\left(\frac{r'}{r}\right)^2\right], \tag{13.65}$$

$$(r'\sqrt{Wv})' = 2r\sqrt{\omega}. \tag{13.66}$$

Here and below the prime on a symbol means differentiation with respect to $\sqrt{W}$, and by m we designate the graviton mass. The functions v and ω are linked

with the functions U, V, and W through the following relationships:

$$v = WUV^{-1}, \quad \omega = UV. \tag{13.67}$$

In the approximation where

$$2M \sim \sqrt{W} \ll \frac{1}{m} \text{ and } rm \ll 1, \tag{13.68}$$

where M is the mass of the source of the gravitational field, we have

$$\sqrt{v} - 2m^2M^3 \ln\left(1 + \frac{\sqrt{v}}{2m^2M^3}\right) = \sqrt{W} - 2M, \tag{13.69}$$

$$\omega = \left(1 + \frac{2m^2M^3}{\sqrt{v}}\right)^2. \tag{13.70}$$

If the graviton mass is nonzero in region

$$\eta \equiv \frac{\sqrt{v}}{2m^2M^3} \ll 1, \tag{13.71}$$

in the vicinity of the Schwarzschild sphere Eq. (13.69) yields

$$v = 4m^2M^3 (\sqrt{W} - 2M). \tag{13.72}$$

Combining (13.70) and (13.72) with definition (13.67), we find that in the vicinity of the Schwarzschild sphere

$$U = m^2M^2, \quad V = M(\sqrt{W} - 2M)^{-1}. \tag{13.73}$$

On the basis of this we conclude that the determinant $g = -UVW^2 \sin^2\theta$ and the invariants $g_{ik}\gamma^{ik}$, $R_{ik}\gamma^{ik}$, and $R_{ikpq}\gamma^{ip}\gamma^{kq}$ have a true singularity on the Schwarzschild sphere that cannot be removed by choosing an appropriate reference frame. This implies the metric g_{ik} inside the Schwarzschild sphere does not correspond to any physical gravitational field.

If we go over to the synchronous reference frame of freely falling test particles with a zero velocity at infinity via the transformation formulas (Vlasov and Logunov, 1988a, 1988b)

$$\tau = t + \int d\sqrt{W} \sqrt{\frac{V}{U}(1-U)}, \tag{13.74a}$$

$$R = t + \int d\sqrt{W} \sqrt{\frac{V}{U(1-U)}}, \tag{13.74b}$$

we find that

$$ds^2 = d\tau^2 - (1-U)\, dR^2 - W(d\theta^2 + \sin^2\theta\, d\varphi^2), \tag{13.75}$$

where U must be expressed in terms of R and τ.

Equation (13.75) implies that for solution (12.75) the singularity on the Schwarzschild sphere disappears. From this the adherent to GR concludes that the falling observer freely crosses the Schwarzschild sphere, which means that gravitational collapse occurs and a black hole is formed.

As follows from metric (13.63) and the transformation formulas (13.74a) and (13.74b), the radial velocity of a particle falling along the Schwarzschild radius is given by the formula

$$\frac{d\sqrt{W}}{d\tau} = -\frac{\sqrt{1-U}}{\sqrt{UV}}. \tag{13.76}$$

In GR, $UV = 1$ and $U = 1 - 2M/\sqrt{W}$, which implies that the velocity of the particle on the Schwarzschild sphere is finite. Since the Jacobian of the transformation is also finite, one is able in GR to expand the applicability region for Eq. (13.76) to the singularity in the curvature invariant $R_{ikpq}R^{ikpq}$.

In RTG with a massive graviton, the situation is entirely different. Substituting (13.73) into (13.76), we obtain

$$\frac{d\sqrt{W}}{d\tau} = -\frac{1}{mM}\sqrt{\frac{\sqrt{W}-2M}{M}}. \tag{13.77}$$

From this it immediately follows that point $\sqrt{w} = 2M$ is the turning point for the radial motion of matter.

Subtracting (13.74a) from (13.74b) and allowing for (13.73) readily leads us to the following relationship:

$$\sqrt{W} = 2M + \frac{(R-\tau)^2}{4m^2M^3},$$

which suggests that $\sqrt{W} > 2M$.

Thus, the fact that the graviton has a nonzero mass, irrespective of its magnitude, leads to repulsion of particles of matter from the Schwarzschild sphere. In view of the singularity in solution (13.73) on the Schwarzschild sphere it follows that in RTG there can be no spherically symmetric objects, either static or nonstatic, with a radius equal to or less than the gravitational radius, which also means that there are no black holes.

Chapter 14. The Gravitational Field of a Nonstatic Spherically Symmetric Object in RTG. Birkhoff's Theorem

In GR it is proved that the exterior gravitational field generated by a nonstatic spherically symmetric object is reduced to the static gravitational field specified by the Schwarzschild metric (12.76). This assertion has been substantiated by Birkhoff, 1923. However, as noted in Chapter 12, the Schwarzschild metric does not satisfy the RTG equations, which forces us to prove a similar theorem within the framework of RTG. Following Vlasov and Logunov, 1985b, we will now demonstrate that the exterior gravitational field of a nonstatic spherically symmetric object is static in RTG.

Suppose that the line element is given by (13.1). Then the functions $\omega(\tau, R)$ and $B(\tau, R)$ in the exterior of the object considered satisfy the following Hilbert-Einstein equations:

$$e^{\omega(\tau, R)} = \frac{1}{1+f(R)}\left(\frac{\partial B}{\partial R}\right)^2, \quad \left(\frac{\partial B}{\partial \tau}\right)^2 = f(R) + \frac{2m}{B}, \tag{14.1}$$

where $f(R) > -1$ is an arbitrary function of variable R, and m a positive constant.

Note that the collection of variables $(\tau, R, \theta, \varphi)$ used in representation (13.1) for the line element ds^2 coincides with the collection of comoving coordinates $\xi^i = (\tau, R, \theta, \varphi)$ (see Chapter 13).

To find the solution that satisfy not only Eqs. (14.1) but also RTG equations of the form (8.37),

$$D_k(\sqrt{-g}\, g^{ik}) = 0, \tag{14.2}$$

we must transfer from the comoving coordinates to coordinates $x^i = x^i(t, r, \theta, \varphi)$ via the formulas

$$t = t(\tau, R), \quad r = r(\tau, R) \tag{14.3}$$

and write the line element ds^2 in the form

$$ds^2 = g_{tt}dt^2 + 2g_{tr}dt\, dr + g_{rr}dr^2 - B^2(d\theta^2 + \sin^2\theta\, d\varphi^2). \tag{14.4}$$

Bearing in mind the tensor transformation law and (14.3), we can establish the relationship existing between the metric coefficients of representations (13.1) and (14.4):

$$\begin{gathered} g^{tt} = \left(\frac{\partial t}{\partial\tau}\right)^2 - \left(\frac{\partial t}{\partial R}\right)^2 e^{-\omega}, \quad g^{tr} = \frac{\partial t}{\partial\tau}\frac{\partial r}{\partial\tau} - \frac{\partial t}{\partial R}\frac{\partial r}{\partial R} e^{-\omega}, \\ g^{rr} = \left(\frac{\partial r}{\partial\tau}\right)^2 - \left(\frac{\partial r}{\partial R}\right)^2 e^{-\omega}, \quad g^{\theta\theta} = -\frac{1}{B^2}, \quad g^{\varphi\varphi} = -\frac{1}{B^2\sin^2\theta}. \end{gathered} \tag{14.5}$$

Equations (14.2) then yield

$$\frac{\partial}{\partial\tau}\left[B^2 e^{\omega/2}\frac{\partial t}{\partial\tau}\right] = \frac{\partial}{\partial R}\left[B^2 e^{-\omega/2}\frac{\partial t}{\partial R}\right], \tag{14.6}$$

$$\frac{\partial}{\partial\tau}\left[B^2 e^{\omega/2}\frac{\partial r}{\partial\tau}\right] = \frac{\partial}{\partial R}\left[B^2 e^{-\omega/2}\frac{\partial r}{\partial R}\right] - 2re^{\omega/2}. \tag{14.7}$$

Note that at $f(R) = 0$ the system of equations (14.1), (14.6), (14.7) coincides with the system (13.9), (13.26), (13.27).

Now let us find the solutions to Eqs. (14.6) and (14.7) for all values of f greater than -1. We will seek a metric g^{ik} (14.5) independent of variable t:

$$\frac{\partial}{\partial t} g^{ik}(x) = 0. \tag{14.8}$$

If in (14.8) we differentiate with respect to τ and R, we get

$$\frac{\partial g^{ik}}{\partial\tau}\frac{\partial r}{\partial R} = \frac{\partial g^{ik}}{\partial R}\frac{\partial r}{\partial\tau}. \tag{14.9}$$

For $(i, k) = (\theta, \theta)$ this yields

$$\frac{\partial B}{\partial\tau}\frac{\partial r}{\partial R} = \frac{\partial B}{\partial R}\frac{\partial r}{\partial\tau},$$

which implies that $r = r(B)$. Therefore, we seek the solution to Eqs. (14.7) in the form $r = r(B)$. We then have

$$\frac{\partial^2 r}{\partial B^2}(B^2 - 2mB) + \frac{\partial r}{\partial B}(2B - 2m) - 2r = 0, \quad B > 2m.$$

This equation has a unique regular solution (Fock, 1939, 1959, and Vlasov and Logunov, 1985b)

$$r = B - m, \quad r \geqslant m. \tag{14.10}$$

Since in view of (14.8) g^{tt} must depend on r, assuming that $g^{tr} = 0$ and allowing for (14.5) and (14.10), we arrive at the following relationships for $\partial t/\partial\tau$

and $\partial t/\partial R$:

$$\left(\frac{\partial t}{\partial \tau}\right)^2-\left(\frac{\partial t}{\partial R}\right)^2\left(\frac{\partial B}{\partial R}\right)^{-2}(1+f)=g^{tt}(r)\equiv H(B),$$
$$\frac{\partial t}{\partial \tau}\frac{\partial B}{\partial \tau}-\frac{\partial t}{\partial R}\left(\frac{\partial B}{\partial R}\right)^{-1}(1+f)=0. \tag{14.11}$$

This implies that

$$\frac{\partial t}{\partial \tau}=\sqrt{1+f}\,\Psi(B), \quad \frac{\partial t}{\partial R}=\frac{\partial B}{\partial \tau}\frac{\partial B}{\partial R}\frac{\Psi(B)}{\sqrt{1+f}}, \tag{14.12}$$

where

$$\Psi(B)=\frac{H^{1/2}(B)}{1-2m/B}.$$

The simultaneity condition for system (14.12),

$$\frac{\partial}{\partial R}\left(\frac{\partial t}{\partial \tau}\right)=\frac{\partial}{\partial \tau}\left(\frac{\partial t}{\partial R}\right),$$

enables us to find the function $\Psi(B)$, for which we have the following equation:

$$\frac{\partial \Psi}{\partial B}\left(1-\frac{2m}{B}\right)+\Psi\frac{2m}{B^2}=0.$$

Taking for the solution to the equation the function

$$\Psi(B)=(1-2m/B)^{-1}, \tag{14.13}$$

we find that

$$H(B)=1.$$

Then Eqs. (14.12) yield the following system of equations for t:

$$\frac{\partial t}{\partial \tau}=\sqrt{1+f}\left(1-\frac{2m}{B}\right)^{-1},$$
$$\frac{\partial t}{\partial R}=\frac{\partial B}{\partial \tau}\frac{\partial B}{\partial R}\left(1-\frac{2m}{B}\right)^{-1}\frac{1}{\sqrt{1+f}}. \tag{14.14}$$

Integration yields

$$t(B,R)=-\sqrt{1+f}\int^{B} dB'\,[f+2m/B']^{-1/2}(1-2m/B')^{-1}. \tag{14.15}$$

For f positive we have

$$t(B,R)=2m\ln\frac{\tanh(\eta/2)+\tanh y}{\tanh(\eta/2)-\tanh y}$$
$$+2m\coth y\left[(\eta-\sinh\eta)\frac{1}{2\sinh^2 y}-\eta\right],$$

where

$$f(R)\equiv\sinh^2 y(R), \quad \cosh\eta\equiv f\frac{B}{m}+1,$$
$$\tanh^2\frac{\eta}{2}-\tanh^2 y=\left(\frac{B}{2m}-1\right)\left(1+\frac{fB}{2m}\right)^{-1}\frac{f}{1+f}.$$

For $f=0$ the expression for t simplifies and assumes the form (13.28). The explicit expression for $t(B,R)$ when f is negative will not be given here due to its complexity.

Substituting (14.10) and (14.14) into (14.5), we arrive at the sought exterior solution to the system of RTG equations:

$$ds^2 = \frac{r-m}{r+m}\,dt^2 - \frac{r+m}{r-m}\,dr^2 - (r+m)^2\,(d\theta^2 + \sin^2\theta\,d\varphi^2). \tag{14.16}$$

But this solution corresponds to the case of a static spherically symmetric object (see Chapter 12). Hence, Birkhoff's theorem holds true in RTG, or a nonstatic spherically symmetric source generates a static gravitational field in the exterior of the source.

Chapter 15. Gravitational Waves

One of the most important problems in the theory of gravitation is the problem of generation and detection of gravitational waves. A comprehensive theoretical investigation of this problem encounters a number of difficulties primarily linked with the strong nonlinearity of the field equations, and only in the weak-field approximation has it been possible to carry out a consistent investigation. Nobody has yet discovered gravitational waves. A natural assumption is that in view of the extremely low intensity of gravitational waves the linearized field equations provide an ideal tool in the hands of researchers studying gravitational waves coming from sources observable in our Universe.

Let us assume that in the entire space-time, including the region occupied by the source, the gravitationl field $\Phi^{mn}(x)$ is weak:

$$|\,\Phi^{mn}(x)\,| \ll 1. \tag{15.1}$$

As shown in Chapter 11, in Cartesian coordinates and in the weak-field approximation the generalized RTG system of equations (11.14), (11.15) can be represented in the form

$$(\Box + m^2)\,\Phi^{mn} = 16\pi \overset{(0)}{T}{}^{mn}, \tag{15.2}$$

$$\partial_m \Phi^{mn} = 0. \tag{15.3}$$

If in reality the graviton mass is very small but finite, the contribution of this mass will be a quantity comparable with second-order or higher-order perturbation terms and, hence, will have no effect on the emission of gravitational waves in the linear approximation. Only on the cosmological scale can this term manifest itself.

In this chapter we wish to compare the results obtained in GR and RTG. Therefore, we consider the system of equations

$$\Box \Phi^{mn} = 16\pi \overset{(0)}{T}{}^{mn}, \tag{15.4}$$

$$\partial_m \Phi^{mn} = 0, \tag{15.5}$$

which follows from (15.2) and (15.3) if we assume that the graviton mass is exactly zero. The quantity $\overset{(0)}{T}{}^{mn}$ on the right-hand side of (15.4) is obtained from the Hilbert energy-momentum tensor for matter after we replace g^{mn} with γ^{mn} and ∇_n with ∂_n in the latter. The covariant conservation law (8.12) assumes the following form in the weak-field approximation:

$$\partial_m \overset{(0)}{T}{}^{mn} = 0. \tag{15.6}$$

Let us use the standard method of solving Eqs. (15.4). We write the tensors $\Phi^{mn}(\mathbf{r}, t)$ and $\overset{(0)}{T}{}^{mn}(\mathbf{r}, t)$ in terms of temporal Fourier integrals:

$$\Phi^{mn}(\mathbf{r}, t) = \int\limits_{-\infty}^{\infty} e^{-i\omega t}\Phi^{mn}(\mathbf{r}, \omega)\, d\omega, \tag{15.7}$$

$$\overset{(0)}{T}{}^{mn}(\mathbf{r}, t) = \int\limits_{-\infty}^{\infty} e^{-i\omega t}\overset{(0)}{T}{}^{mn}(\mathbf{r}, \omega)\, d\omega. \tag{15.8}$$

Since $\Phi^{mn}(\mathbf{r}, t)$ and $\overset{(0)}{T}{}^{mn}(\mathbf{r}, t)$ are real, the integral representations (15.7) and (15.8) yield

$$(\Phi^{mn}(\mathbf{r}, \omega))^* = \Phi^{mn}(\mathbf{r}, -\omega),$$
$$(\overset{(0)}{T}{}^{mn}(\mathbf{r}, \omega))^* = \overset{(0)}{T}{}^{mn}(\mathbf{r}, -\omega).$$

Substituting (15.7) and (15.8) into (15.4), we arrive at an equation,

$$(\nabla^2 + \omega^2)\,\Phi^{mn}(\mathbf{r}, \omega) = -16\pi\overset{(0)}{T}{}^{mn}(\mathbf{r}, \omega), \tag{15.9}$$

whose solution is well known:

$$\Phi^{mn}(\mathbf{r}, \omega) = 4\int \frac{e^{i\omega R}}{R}\overset{(0)}{T}{}^{mn}(\mathbf{r}', \omega)\, d^3r', \tag{15.10}$$

where $R = |\mathbf{r}' - \mathbf{r}|$. For the Fourier transforms $\Phi^{mn}(\mathbf{r}, \omega)$ and $\overset{(0)}{T}{}^{mn}(\mathbf{r}, \omega)$, Eqs. (15.5) and (15.6) yield

$$i\omega\Phi^{0n}(\mathbf{r}, \omega) = \partial_\alpha\Phi^{\alpha n}(\mathbf{r}, \omega), \tag{15.11}$$

$$i\omega\overset{(0)}{T}{}^{0n}(\mathbf{r}, \omega) = \partial_\alpha\overset{(0)}{T}{}^{\alpha n}(\mathbf{r}, \omega). \tag{15.12}$$

Here and in what follows the Greek indices assume values 1, 2, and 3. On the basis of (15.11) we can easily express $\Phi^{0n}(\mathbf{r}, \omega)$ in terms of the spatial Fourier transforms $\Phi^{\alpha\beta}(\mathbf{r}, \omega)$ thus:

$$\Phi^{00}(\mathbf{r}, \omega) = -\frac{1}{\omega^2}\,\partial_\alpha\partial_\beta\Phi^{\alpha\beta}(\mathbf{r}, \omega), \tag{15.13}$$

$$\Phi^{0\alpha}(\mathbf{r}, \omega) = -\frac{i}{\omega}\,\partial_\beta\Phi^{\alpha\beta}(\mathbf{r}, \omega). \tag{15.14}$$

Thus, the solution (15.10) to the RTG system of equations contains only six independent Fourier transforms.

The spatial Fourier transforms $\Phi^{\alpha\beta}(\mathbf{r}, \omega)$ can be written in a form that will later enable us to demonstrate the quadrupole nature of $\Phi^{\alpha\beta}(\mathbf{r}, t)$. Allowing for (15.12), we can write formula (15.10) for the spatial Fourier transforms as follows:

$$\begin{aligned}\Phi^{\alpha\beta}(\mathbf{r}, \omega) = -2\omega^2 \Big\{ &\int \frac{e^{i\omega R}}{R}\overset{(0)}{T}{}^{00}(\mathbf{r}', \omega)\, x'^\alpha x'^\beta\, d^3r' \\ &+ \frac{2i}{\omega}\,\partial_\sigma \int \frac{e^{i\omega R}}{R}\overset{(0)}{T}{}^{0\sigma}(\mathbf{r}', \omega)\, x'^\alpha x'^\beta\, d^3r' \\ &- \frac{1}{\omega^2}\,\partial_\sigma\partial_\tau \int \frac{e^{i\omega R}}{R}\overset{(0)}{T}{}^{\sigma\tau}(\mathbf{r}', \omega)\, x'^\alpha x'^\beta\, d^3r' \Big\}. \end{aligned} \tag{15.15}$$

Now we can take advantage of the arbitrariness in the solution to the system of equations (15.4), (15.5). In the weak-field approximation, $\overset{(0)}{T}{}^{mn}(\mathbf{r}, t)$ is independent of $\Phi^{mn}(\mathbf{r}, t)$; hence, if $\Phi^{mn}(\mathbf{r}, t)$ is a solution to the above-mentioned system of equations, so is the function $\Phi'^{mn}(\mathbf{r}, t)$ specified thus:

$$\Phi'^{mn}(\mathbf{r}, t) = \Phi^{mn}(\mathbf{r}, t) + \partial^m a^n(\mathbf{r}, t) + \partial^n a^m(\mathbf{r}, t) - \gamma^{mn}\,\partial_k a^k(\mathbf{r}, t), \quad (15.16)$$

where the 4-vector $a^n(\mathbf{r}, t)$ satisfies the equation

$$\Box a^n(\mathbf{r}, t) = 0. \quad (15.17)$$

It is appropriate to note at this point that (15.16) is a supercoordinate gauge transformation and has no connection with coordinate transformations.

Aside from (15.17), we must impose a condition on $a^n(\mathbf{r}, t)$ that will guarantee the weakness of field $\Phi'^{mn}(\mathbf{r}, t)$. This means that for $\partial^m a^n(\mathbf{r}, t)$ the following inequality must hold true:

$$|\,\partial^m a^n(\mathbf{r}, t)\,| \ll 1. \quad (15.18)$$

Now in the weak-field approximation observables may be calculated on an equal basis using $\Phi^{mn}(\mathbf{r}, t)$ or using $\Phi'^{mn}(\mathbf{r}, t)$. Employing (15.16) and (15.17), we arrive at the following formulas for the Fourier transforms:

$$\Phi'^{00}(\mathbf{r}, \omega) = \Phi^{00}(\mathbf{r}, \omega) - i\omega a^0(\mathbf{r}, \omega) - \partial_\alpha a^\alpha(\mathbf{r}, \omega), \quad (15.19)$$

$$\Phi'^{0\alpha}(\mathbf{r}, \omega) = \Phi^{0\alpha}(\mathbf{r}, \omega) - i\omega a^\alpha(\mathbf{r}, \omega) + \partial^\alpha a^0(\mathbf{r}, \omega), \quad (15.20)$$

$$\Phi'^{\alpha\beta}(\mathbf{r}, \omega) = \Phi^{\alpha\beta}(\mathbf{r}, \omega) + \partial^\alpha a^\beta(\mathbf{r}, \omega) + \partial^\beta a^\alpha(\mathbf{r}, \omega) - \gamma^{\alpha\beta}(\partial_\sigma a^\sigma(\mathbf{r}, \omega) - i\omega a^0(\mathbf{r}, \omega)), \quad (15.21)$$

$$(\omega^2 - \partial_\alpha\partial^\alpha)\, a^n(\mathbf{r}, \omega) = 0. \quad (15.22)$$

Let us select the 4-vector $a^n(\mathbf{r}, \omega)$ such that the $\Phi'^{0\alpha}(\mathbf{r}, \omega)$ ($\alpha = 1, 2, 3$) and the trace of $\Phi'^{mn}(\mathbf{r}, \omega)$, equal to $\Phi'^0_0(\mathbf{r}, \omega) + \Phi'^\alpha_\alpha(\mathbf{r}, \omega)$, vanish. Conditions of this type imposed on the field $\Phi'^{mn}(\mathbf{r}, \omega)$ are known as TT gauge.

On the basis of (15.13), (15.14), and (15.19)-(15.22) one can easily show that the field $\Phi'^{mn}(\mathbf{r}, \omega)$ will satisfy the TT gauge conditions if $a^0(\mathbf{r}, \omega)$ and $a^\alpha(\mathbf{r}, \omega)$ are chosen as follows:

$$a^0(\mathbf{r}, \omega) = -\frac{i}{2\omega}\left[\Phi^{00}(\mathbf{r}, \omega) - \frac{1}{2}\Phi^n_n(\mathbf{r}, \omega)\right], \quad (15.23)$$

$$a^\alpha(\mathbf{r}, \omega) = -\frac{i}{\omega}\Phi^{0\alpha}(\mathbf{r}, \omega) - \frac{1}{2\omega^2}\partial^\alpha\left[\Phi^{00}(\mathbf{r}, \omega) - \frac{1}{2}\Phi^n_n(\mathbf{r}, \omega)\right]. \quad (15.24)$$

Combining (15.23), (15.24), and (15.9) with (15.21), we arrive at the following expression for $\Phi'^{\alpha\beta}(\mathbf{r}, \omega)$ outside matter:

$$\Phi'^{\alpha\beta}(\mathbf{r}, \omega) = S^{\alpha\beta}(\mathbf{r}, \omega) - \frac{1}{\omega^2}(\partial^\alpha\partial_\sigma S^{\sigma\beta}(\mathbf{r}, \omega) + \partial^\beta\partial_\sigma S^{\alpha\sigma}(\mathbf{r}, \omega)) + \frac{1}{2\omega^2}\gamma^{\alpha\beta}\partial_\sigma\partial_\tau S^{\sigma\tau}(\mathbf{r}, \omega) + \frac{1}{2\omega^4}\partial^\alpha\partial^\beta\partial_\sigma\partial_\tau S^{\sigma\tau}(\mathbf{r}, \omega), \quad (15.25)$$

where we have introduced the notation

$$S^{\alpha\beta}(\mathbf{r}, \omega) = \Phi^{\alpha\beta}(\mathbf{r}, \omega) - \frac{1}{3}\gamma^{\alpha\beta}\Phi^\sigma_\sigma(\mathbf{r}, \omega). \quad (15.26)$$

Here $\gamma^{\alpha\beta}$ constitutes the spatial part of the Minkowski metric with -1 elements on the principal diagonal. Note the $S^{\alpha\beta}(\mathbf{r}, \omega)$ is a traceless tensor, that is,

$$\gamma_{\alpha\beta}S^{\alpha\beta}(\mathbf{r}, \omega) = 0. \quad (15.27)$$

Now let us devote more attention to solution (15.15). Expanding R^{-1} in powers of r^{-1}, where r is the distance from the source's center to the point where the field is detected, and assuming that the linear dimensions of the source are much smaller than r, we find that (15.15) yields

$$\Phi^{\alpha\beta}(\mathbf{r},\ \omega) = -\frac{2\omega^2}{r}\int e^{i\omega R}x'^{\alpha}x'^{\beta}[\overset{(0)}{T}{}^{00}(\mathbf{r}',\ \omega) + 2e_\sigma\overset{(0)}{T}{}^{0\sigma}(\mathbf{r}',\ \omega) + e_\sigma e_\tau\overset{(0)}{T}{}^{\sigma\tau}(\mathbf{r}',\ \omega)]\,d^3r', \tag{15.28}$$

with $e^\sigma = x^\sigma/r$ and $e_\sigma e^\sigma = -1$. In going on from (15.15) to (15.28) we have dropped the nonwave terms, which fall off faster than r^{-1}. Substituting (15.28) into (15.26), we get

$$S^{\alpha\beta}(\mathbf{r},\ \omega) = -\frac{2\omega^2}{r}\int e^{i\omega R}\left(x'^{\alpha}x'^{\beta} - \frac{1}{3}\gamma^{\alpha\beta}x'_\sigma x'^{\sigma}\right) \times [\overset{(0)}{T}{}^{00}(\mathbf{r}',\ \omega) + 2e_\sigma\overset{(0)}{T}{}^{0\sigma}(\mathbf{r}',\ \omega) + e_\sigma e_\tau\overset{(0)}{T}{}^{\sigma\tau}(\mathbf{r}',\ \omega)]\,d^3r'. \tag{15.29}$$

It can easily be demonstrated that, to within terms of the order of r^{-2},

$$\partial_\sigma S^{\alpha\beta}(\mathbf{r},\ \omega) = -i\omega e_\sigma S^{\alpha\beta}(\mathbf{r},\ \omega),$$
$$\partial^\sigma S^{\alpha\beta}(\mathbf{r},\ \omega) = -i\omega e^\sigma S^{\alpha\beta}(\mathbf{r},\ \omega).$$

This enables writing (15.25) thus:

$$\Phi'^{\alpha\beta}(\mathbf{r},\ \omega) = S^{\alpha\beta}(\mathbf{r},\ \omega) + e^\alpha e_\sigma S^{\sigma\beta}(\mathbf{r},\ \omega) + e^\beta e_\sigma S^{\sigma\alpha}(\mathbf{r},\ \omega) - \frac{1}{2}\gamma^{\alpha\beta}e_\sigma e_\tau S^{\sigma\tau}(\mathbf{r},\ \omega) + \frac{1}{2}e^\alpha e^\beta e_\sigma e_\tau S^{\sigma\tau}(\mathbf{r},\ \omega). \tag{15.30}$$

Introducing the projection operators

$$P^\alpha_\beta = \delta^\alpha_\beta + e^\alpha e_\beta \tag{15.31}$$

with

$$P^\alpha_\alpha = 2,\quad P^\alpha_\sigma P^\sigma_\beta = P^\alpha_\beta, \tag{15.32}$$

we can represent (15.30) in the compact form

$$\Phi'^{\alpha\beta}(\mathbf{r},\ \omega) = \left(P^\alpha_\tau P^\beta_\sigma - \frac{1}{2}P^{\alpha\beta}P_{\tau\sigma}\right)S^{\sigma\tau}(\mathbf{r},\ \omega). \tag{15.33}$$

Applying the Fourier transformation to both sides of (15.33), we get

$$\Phi'^{\alpha\beta}(\mathbf{r},\ t) = \left(P^\alpha_\tau P^\beta_\sigma - \frac{1}{2}P^{\alpha\beta}P_{\tau\sigma}\right)S^{\sigma\tau}(\mathbf{r},\ t), \tag{15.34}$$

where, in view of (15.29),

$$S^{\sigma\tau}(\mathbf{r},\ t) = \frac{2}{r}\frac{d^2}{dt^2}\int\left(x'^{\sigma}x'^{\tau} - \frac{1}{3}\gamma^{\sigma\tau}x'_\nu x'^{\nu}\right) \times [\overset{(0)}{T}{}^{00}(\mathbf{r}',\ t') + e_\alpha e_\beta\overset{(0)}{T}{}^{\alpha\beta}(\mathbf{r}',\ t')]_{\text{ret}}\,d^3r'. \tag{15.35}$$

Here and in what follows the subscript "ret" on the integrand means that the expression inside the square brackets must be taken at a retarded time $t' = t - R/c$.

Let us define the traceless tensor of the generalized quadrupole moment thus:

$$\mathcal{D}^{\alpha\beta} = D^{\alpha\beta} + 2e_\sigma D^{\alpha\beta\sigma} + e_\sigma e_\tau D^{\alpha\beta\sigma\tau}, \tag{15.36}$$

Let us now establish the transformation properties of $h^{\alpha\beta}$ under rotation of the three-dimensional space about the z axis through an angle θ. Since the nonzero elements in the matrix representing a rotation about the z axis are

$$\Omega_1^1 = \cos\theta, \quad \Omega_1^{\ 2} = \sin\theta, \quad \Omega_2^{\ 1} = -\sin\theta, \quad \Omega_2^2 = \cos\theta, \quad \Omega_3^3 = 1,$$

from (15.45) we find that

$$\begin{gathered} h'^{13} = h'^{23} = h'^{33} = 0, \\ h'^{11} = -h'^{22} = h^{11}\cos 2\theta - h^{12}\sin 2\theta, \\ h'^{12} = h^{11}\sin 2\theta + h^{12}\cos 2\theta. \end{gathered} \tag{15.46}$$

Since outside matter $h^{\alpha\beta}(\mathbf{r}, t)$ satisfies a homogeneous linear equation, any linear combination of components of $h^{\alpha\beta}$ also satisfies this equation. Let us take the following combinations:

$$h_{\pm} = h^{11} \pm ih^{12}.$$

For these (15.46) gives the following transformation law:

$$h'_{\pm} = e^{\pm 2i\theta}h_{\pm}. \tag{15.47}$$

As is well known, if under a rotation of the three-dimensional space through an angle θ about the direction of propagation of a wave a wave function Ψ is transformed as

$$\Psi' = e^{i\lambda\theta}\Psi,$$

then Ψ is an eigenfunction of the helicity operator $\hat{I}_\lambda$ with an eigenvalue equal to λ.

Thus, (15.47) implies that $h_{\pm}$ are the eigenfunctions of operator $\hat{I}_\lambda$ that describe the states of the gravitational field with helicities $\lambda = \pm 2$, respectively. The states of the gravitational field with helicities $\lambda = \pm 1$ and $\lambda = 0$ have no physical meaning since the corresponding eigenfunctions can always be made equal to zero through appropriate gauge transformations.

Thus, the supercoordinate transformations (15.16) have excluded the nonphysical components of Φ^{mn}. The gravitational field proves to be a spin-2 field with helicities ± 2. Only physical components of this field will enter into the expression for the gravitational-wave flux, and the metric tensor for the effective Riemann space-time is constructed on the basis of (8.1) from the physical components of the field. It is this feature that fixes the geometry of the Riemann space-time.

Now, following Einstein, 1918b, we will calculate the intensity of gravitational waves in the GR framework. The method of calculating the intensity suggested by Einstein and the various modifications have gained wide acceptance and are given in many articles and monographs. In this chapter we employ the variant used in Landau and Lifshitz, 1975. As is well known, the Hilbert-Einstein equations lead to the following differential conservation law:

$$\partial_n \left[-g\left(T^{mn} + \tau^{mn}\right)\right] = 0, \tag{15.48}$$

where $\tau^{mn} = \tau^{nm}$ is the energy-momentum pseudotensor of the gravitational field. Integrating (15.48) over a sufficiently large volume and assuming that there is no transport of matter through the surface bounding the integration volume, we obtain

$$\frac{d}{dt}\int(-g)\left[T^{0m} + \tau^{0m}\right]dV = -\oint(-g)\,\tau^{\alpha m}\,dS_\alpha. \tag{15.49}$$

According to Einstein, 1918b, the right-hand side of (15.49) at $m = 0$ is "for certain the loss of energy by the material system" and, hence,

$$\frac{dE}{dt} = -\oint (-g)\,\tau^{0\alpha}\,dS_\alpha. \tag{15.50}$$

Then the "energy flux" carried by the gravitational waves through the elemental surface area dS_α is given by the formula

$$dI = (-g)\,\tau^{0\alpha}\,dS_\alpha. \tag{15.51}$$

If for the surface of integration we take a sphere of radius r (the $dS_\alpha = -r^2 e_\alpha\,d\Omega$), then for the intensity of gravitational waves per unit solid angle we have

$$\frac{dI}{d\Omega} = -r^2(-g)\,\tau^{0\alpha}e_\alpha, \tag{15.52}$$

where $d\Omega$ is the unit solid angle. Calculating $(-g)\,\tau^{0\alpha}$ from, say, the pseudotensor given in Landau and Lifshitz, 1975 (p. 282), we find that in the weak-field approximation combined with (15.41) and (15.42) and the TT gauge conditions,

$$(-g)\,\tau^{0\alpha} = \frac{e^\alpha}{32\pi}\left(\frac{dh^{\mu\nu}}{dt}\right)\left(\frac{dh_{\mu\nu}}{dt}\right). \tag{15.53}$$

This leads to the following transformation of formula (15.52) for the intensity of gravitational waves per unit solid angle:

$$\frac{dI}{d\Omega} = \frac{r^2}{32\pi}\left(\frac{dh^{\mu\nu}}{dt}\right)\left(\frac{dh_{\mu\nu}}{dt}\right). \tag{15.54}$$

Substitution of (15.44) into (15.54) yields

$$\frac{dI}{d\Omega} = \frac{1}{36\pi}\left[\frac{1}{4}(e_\alpha e_\beta \dddot{D}^{\alpha\beta})^2 + \frac{1}{2}\dddot{D}^{\alpha\beta}\dddot{D}_{\alpha\beta} + e_\alpha e^\beta \dddot{D}_{\beta\sigma}\dddot{D}^{\sigma\alpha}\right]. \tag{15.55}$$

Integrating (15.55) with respect to the angular variables and allowing for

$$\int d\Omega\, e_\alpha e_\beta = -\frac{4\pi}{3}\gamma_{\alpha\beta},$$

$$\int d\Omega\, e_\alpha e_\beta e_\sigma e_\tau = \frac{4\pi}{15}(\gamma_{\alpha\beta}\gamma_{\sigma\tau} + \gamma_{\alpha\sigma}\gamma_{\beta\tau} + \gamma_{\alpha\tau}\gamma_{\beta\sigma}),$$

we arrive at the well-known quadrupole formula for the total intensity, which was first established by Einstein, 1918b:

$$I = \frac{1}{45}\dddot{D}_{\alpha\beta}\dddot{D}^{\alpha\beta}, \tag{15.56}$$

which implies that

$$I > 0. \tag{15.57}$$

Formula (15.56) for the intensity of gravitational waves has been derived in GR on the assumption that the x^h coordinates in the entire Riemann space-time are Cartesian. If we were to select other coordinates in GR, we would arrive at an entirely different result, which means that (15.56) is not a corollary of GR. The derivation of formula (15.56) given above in the GR framework is based on the definition of energy flux via (15.52). And the latter contains the quantity $\tau^{0\alpha}$, which is not a tensor.

An analysis conducted in Denisov and Logunov, 1982d, and Vlasov and Denisov, 1982, has shown that, depending on the choice of the system of coordinates, the energy flux (15.52) through every element of a spherical surface of arbitrary radius

r and, hence, the total energy flow (or total intensity) through the entire spherical surface in the course of a finite and fixed time interval may be positive or negative or zero, in contrast to the assertion in Einstein, 1918b.

By selecting an appropriate reference frame in the GR framework and *starting from definition* (15.52), in the weak-field approximation we can derive (Denisov and Logunov, 1982d, and Vlasov and Denisov, 1982) the following formulas for the intensity per unit solid angle and total intensity of gravitational waves:

$$\frac{dI}{d\Omega}=\frac{1-a^2}{36\pi}\left[\frac{1}{4}(e_\alpha e_\beta \dddot{D}^{\alpha\beta})^2+\frac{1}{2}\dddot{D}_{\alpha\beta}\dddot{D}^{\alpha\beta}+e_\alpha e^\beta \dddot{D}_{\beta\sigma}\dddot{D}^{\sigma\alpha}\right], \tag{15.58}$$

$$I=\frac{1-a^2}{45}\dddot{D}_{\alpha\beta}\dddot{D}^{\alpha\beta}, \tag{15.59}$$

with a an arbitrary constant. We see that only when $a=0$ do we get formulas (15.55) and (15.56). If, for one thing, we take a equal to $+1$ or -1, we find that the intensity per unit solid angle $dI/d\Omega$ and the total intensity vanish.

We have thus arrived at the conclusion that an appropriate choice of reference frame may result in the arbitrary sign of $dI/d\Omega$ and I defined in GR, according to Einstein, on the basis of (15.52) or even nullify these two quantities. *This is absurd from the standpoint of physics, since radiation, being a physical reality, cannot be "annihilated" by an appropriate transformation of coordinates.*

In contrast to GR, in the framework of RTG, as we will shortly demonstrate, there are no such difficulties and the formulas (15.55) and (15.56) are strict corollaries of our theoretical ideas.

At the base of intensity calculations we place the covariant RTG conservation law in the (9.23) form:

$$D_m(T^m_n+\overset{(0)}{t}{}^m_{(g)n})=0. \tag{15.60}$$

In Chapter 9 this form of the covariant conservation law for the energy-momentum tensor of matter and gravitational field taken together was shown to be in all respects similar to the covariant conservation law (6.28). We have chosen (15.60) as the form for the conservation law for purely technical reasons. For $\overset{(0)}{t}{}^m_{(g)n}$ we already have the representation (9.13), in which we have isolated the term that is the covariant divergence of the tensor K^{pq}_m, which is antisymmetric in the upper indices, and that, therefore, contributes nothing to (15.60).* If we allow for (9.13), formula (15.60) can be rewritten as follows:

$$D_m(T^m_n+\tau^m_n)=0, \tag{15.61}$$

where

$$\tau^m_n=-\delta^m_n L_g+\frac{1}{16\pi}\left[\widetilde{G}^m_{pq}+\frac{1}{2}\widetilde{g}^{mk}\widetilde{g}_{pq}\widetilde{G}^l_{kl}\right]D_n\widetilde{g}^{pq}. \tag{15.62}$$

In contrast to (15.48), the left-hand side of (15.61) is a true tensor since it is the covariant divergence in the Minkowski metric of the tensor quantities T^m_n and τ^m_n. Hence, an intensity calculation (or calculation of other characteristics of the gravitational field) based on (15.61) will not depend on the choice of the system of coordinates. Since in the Minkowski space-time we can always select the Cartesian (Galilean) system of coordinates, (15.61) yields

$$\partial_m(T^m_n+\tau^m_n)=0. \tag{15.63}$$

* Terms whose divergence vanishes identically in view of their structure lead to no law of conservation.

Thus, in RTG, in contrast to GR, the conservation law (15.63) is automatically written in terms of Cartesian coordinates. Integrating (15.63) over a sufficiently large volume and assuming that matter is not transported through the surface bounding the integration volume, we get

$$\partial_0 \int (T_n^0 + \tau_n^0)\, dV = -\oint \tau_n^\alpha\, dS_\alpha. \tag{15.64}$$

Since at $n = 0$ the left-hand side of (15.64) is the amount of energy lost by the system, the energy flux carried by gravitational waves through the elemental surface area dS_α is

$$dI = \tau_0^\alpha\, dS_\alpha. \tag{15.65}$$

If we select the surface of a sphere of radius r as the integration surface, we arrive at the following formula for the intensity per unit solid angle:

$$\frac{dI}{d\Omega} = -r^2 \tau_0^\alpha\, dS_\alpha. \tag{15.66}$$

To find the explicit form of (15.66), we must calculate τ_0^α in the weak-field approximation. Writing (15.62) in terms of Cartesian coordinates and employing the expansions (15.41) and (15.42) in the weak-field approximation combined with the TT gauge conditions, we get

$$\tau_0^\alpha = -\frac{1}{16\pi}\, \partial_0 h^{\sigma\tau}\, \partial_\sigma h_\tau^\alpha + \frac{1}{32\pi}\, \partial_0 h^{\sigma\tau} \partial^\alpha h_{\sigma\tau}. \tag{15.67}$$

Since to within terms of the order of $1/r^2$ and higher

$$\partial^\alpha h_{\sigma\tau} = e^\alpha\, \partial_0 h_{\sigma\tau} \tag{15.68}$$

and the TT gauge conditions imply that

$$\partial_\alpha h^{\alpha\beta} = 0,$$

formula (15.67) yields

$$\tau_0^\alpha e_\alpha = -\frac{1}{32\pi}\, \partial_0 h^{\sigma\tau}\, \partial_0 h_{\sigma\tau}. \tag{15.69}$$

Combining this with (15.66), we arrive at the following formula for the intensity per unit solid angle:

$$\frac{dI}{d\Omega} = \frac{r^2}{32\pi}\, \partial_0 h^{\alpha\beta}\, \partial_0 h_{\alpha\beta}.$$

If we combine this with (15.44), we arrive at (15.55) and (15.56):

$$\frac{dI}{\partial\Omega} = \frac{1}{36\pi}\left[\frac{1}{4}(e_\alpha e_\beta \dddot{D}^{\alpha\beta})^2 + \frac{1}{2}\, \dddot{D}^{\alpha\beta} \dddot{D}_{\alpha\beta} + e_\alpha e^\beta \dddot{D}_{\sigma\beta} \dddot{D}^{\sigma\alpha}\right], \tag{15.70}$$

$$I = \frac{1}{45}\, \dddot{D}_{\alpha\beta} \dddot{D}^{\alpha\beta}. \tag{15.71}$$

In RTG the expression for the energy-momentum tensor of the gravitational field in an inertial reference frame formally coincides, when written in terms of Galilean coordinates, with the expression for the GR energy-momentum pseudotensor if for some reason the spatial-temporal coordinates in the latter are declared Cartesian. But in the Riemann space-time, and, therefore, in GR, there can be no system of global Cartesian coordinates.

In the case of a weak field such a formal coincidence enabled Einstein intuitively to find the formula for the intensity of gravitational waves.

Chapter 16. A Homogeneous and Isotropic Universe

In this chapter we consider a homogeneous and isotropic universe. The line element for such a universe will be represented in the following form:

$$ds^2 = U(t, r)\, dt^2 - V(t, r)\, dr^2 - W(t, r)\,(d\theta^2 + \sin^2\theta\, d\varphi^2), \tag{16.1}$$

where t is time and r, θ, and φ are the spherical coordinates in the Minkowski space-time. The functions $U(t, r)$, $V(t, r)$, and $W(t, r)$ for a given distribution of matter must be found by solving the system of equations (8.36), (8.37). In what follows it is convenient to employ the system (8.36), (8.37) in a form in which the Ricci tensor and the energy-momentum tensor are expressed in terms of mixed coordinates:

$$\sqrt{-g}\left(R_n^m - \frac{1}{2}\delta_n^m R\right) = 8\pi T_n^m, \tag{16.2}$$

$$D_m \tilde{g}^{mn} = 0. \tag{16.3}$$

For T_n^m we take the energy-momentum tensor density of a perfect fluid (Fock, 1939, 1959, and Tolman, 1934):

$$T_n^m = \sqrt{-g}\,[(\rho + p)\, u^m u_n - \delta_n^m p], \tag{16.4}$$

where ρ is the density of the fluid, p the isotropic pressure, and u^m the unit 4-vector of velocity.

According to (16.1), the nonzero components of the metric tensor g_{mn} are

$$g_{00} = U(t, r), \quad g_{11} = -V(t, r), \quad g_{22} = -W(t, r), \quad g_{33} = -W(t, r)\sin^2\theta. \tag{16.5}$$

If we recall that $g_{mn}g^{nk} = \delta_m^k$, we can easily find the nonzero components of g^{mn}:

$$g^{00} = \frac{1}{U(t, r)}, \quad g^{11} = -\frac{1}{V(t, r)}, \quad g^{22} = -\frac{1}{W(t, r)}, \quad g^{33} = -\frac{1}{W(t, r)\sin^2\theta}, \tag{16.6}$$

with $g^{mn} = 0$ for $m \neq n$. Let us assume that

$$u^\alpha = 0, \quad \alpha = 1, 2, 3. \tag{16.7}$$

Then, in view of the fact that $g_{mn}u^m u^n = 1$, we find that

$$u^0 = \frac{1}{\sqrt{U}}, \quad u_0 = \sqrt{U}. \tag{16.8}$$

Combining (16.7) and (16.8) with (16.4), we obtain

$$T_0^0 = \sqrt{-g}\,\rho, \quad T_1^1 = T_2^2 = T_3^3 = -\sqrt{-g}\,p, \tag{16.9}$$

with $T^m_n = 0$ for $m \neq n$. If we now take into account (16.5) and (16.6), we find that the nonzero Riemannian connection coefficients are

$$\Gamma^0_{00} = \frac{1}{2U}\frac{\partial U}{\partial t}, \quad \Gamma^1_{00} = \frac{1}{2V}\frac{\partial U}{\partial r}, \quad \Gamma^0_{01} = \frac{1}{2U}\frac{\partial U}{\partial r}, \quad \Gamma^1_{11} = \frac{1}{2V}\frac{\partial V}{\partial r},$$
$$\Gamma^0_{11} = \frac{1}{2U}\frac{\partial V}{\partial t}, \quad \Gamma^1_{01} = \frac{1}{2V}\frac{\partial V}{\partial t}, \quad \Gamma^0_{22} = \frac{1}{2U}\frac{\partial W}{\partial t}, \quad \Gamma^1_{22} = -\frac{1}{2V}\frac{\partial W}{\partial r},$$
$$\Gamma^2_{02} = \frac{1}{2W}\frac{\partial W}{\partial t}, \quad \Gamma^3_{03} = \frac{1}{2W}\frac{\partial W}{\partial t}, \quad \Gamma^2_{12} = \frac{1}{2W}\frac{\partial W}{\partial r}, \quad \Gamma^3_{13} = \frac{1}{2W}\frac{\partial W}{\partial r}, \tag{16.10}$$
$$\Gamma^0_{33} = \Gamma^0_{22}\sin^2\theta, \quad \Gamma^1_{33} = \Gamma^1_{22}\sin^2\theta, \quad \Gamma^2_{33} = -\sin\theta\cos\theta,$$
$$\Gamma^3_{23} = \cot\theta.$$

Using formulas (16.9) and (16.10) in the covariant "conservation" law $\nabla_m T^m_n = 0$, we can easily obtain the equations

$$\frac{1}{\rho+p}\frac{\partial\rho}{\partial t} = -\frac{\partial}{\partial t}(\ln\sqrt{V}\,W), \tag{16.11}$$

$$\frac{1}{\rho+p}\frac{\partial p}{\partial r} = -\frac{\partial}{\partial r}\ln\sqrt{U}, \tag{16.12}$$

which link ρ, p, U, V, and W. Let us start with Eqs. (16.2), (16.3) and establish the relationships that exist between U, V, and W. We write (16.3) in the form

$$\partial_m \tilde{g}^{mn} + \gamma^n_{pm}\tilde{g}^{mp} = 0, \tag{16.13}$$

where γ^n_{pm} are the Christoffel symbols for the Minkowski space-time whose nonzero components are given in (12.4) if spherical coordinates are used. Since $\sqrt{-g} = \sqrt{UV}\,W\sin\theta$ (see (16.5)), we find from the definition $\tilde{g}^{mn} = \sqrt{-g}g^{mn}$ combined with (16.6) that

$$\tilde{g}^{00} = \sqrt{V/U}\,W\sin\theta, \quad \tilde{g}^{11} = -\sqrt{U/V}\,W\sin\theta,$$
$$\tilde{g}^{22} = -\sqrt{UV}\sin\theta, \quad \tilde{g}^{33} = -\sqrt{UV}(\sin\theta)^{-1}. \tag{16.14}$$

Substituting (12.4) and (16.14) into (16.13), we get

$$\frac{\partial}{\partial t}(\sqrt{V/U}\,W) = 0, \tag{16.15}$$

$$\frac{\partial}{\partial r}(\sqrt{U/V}\,W) = 2r\sqrt{UV}. \tag{16.16}$$

Equation (16.15) implies that

$$\sqrt{V/U}\,W = f(r), \tag{16.17}$$

where $f(r)$ is an arbitrary function that does not depend on t. Since in view of (16.17) we have $\sqrt{V}\,W = \sqrt{U}f(r)$, Eq. (16.11) can be written as follows:

$$\frac{1}{\rho+p}\frac{\partial\rho}{\partial t} = -\frac{\partial}{\partial t}\ln\sqrt{U}.$$

Thus, the covariant conservation law $\nabla_m T^m_n = 0$ and the RTG equations (16.2) for ρ, p, U, V, and W yield the following relationships:

$$\frac{1}{\rho+p}\frac{\partial\rho}{\partial t} = -\frac{\partial}{\partial t}\ln\sqrt{U}, \tag{16.18}$$

$$\frac{1}{\rho+p}\frac{\partial p}{\partial r} = -\frac{\partial}{\partial r}\ln\sqrt{U}, \tag{16.19}$$

$$\frac{\partial}{\partial r}(\sqrt{U/V}\,W) = 2r\sqrt{UV}, \tag{16.20}$$

$$\sqrt{V}\,W = \sqrt{U}\,f(r). \tag{16.21}$$

Equations (16.18) and (16.19) imply that for them to be valid simultaneously, the following condition must hold true:

$$\frac{\partial}{\partial r}\left(\frac{1}{\rho+p}\frac{\partial\rho}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{1}{\rho+p}\frac{\partial p}{\partial r}\right). \tag{16.22}$$

Note that condition (16.22) emerged only after Eq. (16.11) was represented in the form (16.18), and this became possible thanks to formula (16.17), which follows from the RTG equations (16.3). This suggests that the differential relationship (16.22), which connects only the characteristics of matter, ρ and p, appears necessarily only in RTG. Let us illustrate this fact by an example. We will see what restrictions are imposed by condition (16.22).

Suppose that the equation of state of matter has the form $\rho(t, r) = p^\alpha(t, r)$, where the exponent α is a number. We assume that both $\rho(t, r)$ and $p(t, r)$ are separable, that is,

$$\rho(t, r) = \rho_1(t)\,\rho_2(r), \quad p(t, r) = p_1(t)\,p_2(r).$$

Then (16.22) yields

$$\frac{dp_1^\alpha}{dt}\frac{dp_2^\alpha}{dr} = p_1^{\alpha-1}p_2^{\alpha-1}\frac{dp_1}{dt}\frac{dp_2}{dr}.$$

This is possible only if $\alpha = 1$ or $\alpha = -1$. The case with $\alpha = -1$ must be excluded since at $\rho_1(t)\,\rho_2(r) = p_1^{-1}(t)\,p_2^{-1}(r)$ Eqs. (16.18) and (16.19) have no solution.

Thus, if the state of matter is such that both the energy density $\rho(t, r)$ and the isotropic pressure $p(t, r)$ are separable and in the equation $\rho = p^\alpha$ the exponent α is not equal to unity, the effective Riemann space-time cannot have a line element in the form (16.1); and vice versa, if the line element is represented by (16.1) and $\rho(t, r)$ and $p(t, r)$ are separable, the exponent α in the equation of state $\rho = p^\alpha$ is necessarily equal to $+1$. If $\alpha = 1$, (16.18) immediately implies that $U(t, r)$ is separable.

Below we consider, following Vlasov and Logunov, 1986b, the model of a homogeneous and isotropic universe in which density ρ and isotropic pressure p are functions of only the time variable in the Minkowski space-time. In this case condition (16.22) is satisfied automatically and (16.19) implies that

$$U = U(t). \tag{16.23}$$

Since U is independent of r, (16.20) and (16.21) yield

$$\frac{\partial}{\partial r}\left(\frac{f}{V}\right) = 2r\sqrt{\frac{V}{U}}.$$

Multiplying this into $(f/V)^{1/2}$, we have

$$\frac{\partial}{\partial r}\left(\frac{f}{V}\right)^{3/2}=3r\left(\frac{f}{U}\right)^{1/2},$$

which yields the following formula for $V(t, r)$:

$$V^{3/2}(t, r)=\sqrt{U(t)}\,f^{3/2}(r)\,[Z(r)+\varphi(t)]^{-1}, \tag{16.24}$$

where

$$Z(r)=3\int rf^{1/2}(r)\,dr, \tag{16.25}$$

and $\varphi(t)$ is an arbitrary function of t. Combining (16.21) with (16.24), we get

$$W(t, r)=f^{1/2}(r)\,U^{1/3}(t)\,[Z(r)+\varphi(t)]^{1/3}, \tag{16.26}$$

$$V(t, r)=\frac{f^2(r)\,U(t)}{W^2(t, r)}. \tag{16.27}$$

We have not yet discussed how the Hilbert-Einstein equations (16.2) come into the picture. Let us examine them. Assuming that $m=1$ and $n=0$ and employing (12.11) as a basis combined with (16.9), (16.10), and (16.23), we find that

$$-\frac{\partial^2 W}{\partial t\,\partial r}+\frac{1}{2W}\frac{\partial W}{\partial r}\frac{\partial W}{\partial t}+\frac{1}{2V}\frac{\partial V}{\partial t}\frac{\partial W}{\partial r}=0. \tag{16.28}$$

Dividing this equation by $\partial W/\partial r$ and performing simple manipulations, we get

$$\frac{\partial}{\partial t}\ln\left[\frac{\partial W}{\partial r}\frac{1}{\sqrt{WV}}\right]=0.$$

This yields

$$\frac{\partial W}{\partial r}=c(r)\sqrt{WV}, \tag{16.29}$$

where $c(r)$ is an arbitrary function depending only on r. If we allow for (16.21), Eq. (16.29) can be written thus:

$$\frac{\partial W^{3/2}}{\partial r}=\frac{3}{2}\sqrt{U(t)}\,c(r)\,f(r).$$

Integration yields

$$W^{3/2}(t, r)=\sqrt{U(t)}\left[\frac{3}{2}\int c(r)\,f(r)\,dr+b(t)\right], \tag{16.30}$$

with $b(t)$ an arbitrary function. Comparison of (16.26) with (16.30) shows that the two are valid simultaneously if

$$b(t)=\varphi(t)=0, \tag{16.31}$$

$$\int c(r)\,f(r)\,dr=\frac{2}{3}f^{3/4}(r)\left[3\int r\sqrt{f(r)}\,dr\right]^{1/2}. \tag{16.32}$$

The last formula can be used to determine $c(r)$.

Taking into account (16.31), we can write Eqs. (16.26) and (16.27) as follows:

$$W(t, r)=U^{1/3}(t)\,W_1(r), \tag{16.33}$$

$$V(t, r)=U^{1/3}(t)\,V_1(r), \tag{16.34}$$

where

$$W_1(r)=f^{1/2}(r)\,Z^{1/3}(r), \tag{16.35}$$

$$V_1(r)=f(r)\,Z^{-2/3}(r). \tag{16.36}$$

On the basis of (16.33) and (16.34), we can easily establish that formulas (16.10) lead to the following expressions for the nonzero Riemannian connection coefficients:

$$\begin{gathered}
\Gamma^0_{00}=\frac{1}{2U}\frac{dU}{dt},\quad \Gamma^2_{12}=\Gamma^3_{13}=\frac{1}{2W_1}\frac{dW_1}{dr},\\
\Gamma^0_{11}=\frac{V_1}{6U^{5/3}}\frac{dU}{dt},\quad \Gamma^2_{33}=-\sin\theta\cos\theta,\\
\Gamma^0_{22}=\frac{W_1}{6U^{5/3}}\frac{dU}{dt},\quad \Gamma^1_{11}=\frac{1}{2V_1}\frac{dV_1}{dr},\\
\Gamma^0_{33}=\Gamma^0_{22}\sin^2\theta,\quad \Gamma^1_{22}=-\frac{1}{2V_1}\frac{dW_1}{dr},\\
\Gamma^1_{01}=\Gamma^2_{02}=\Gamma^3_{03}=\frac{1}{6U}\frac{dU}{dt},\quad \Gamma^1_{33}=\Gamma^1_{22}\sin^2\theta,\quad \Gamma^3_{23}=\cot\theta.
\end{gathered}\tag{16.37}$$

Putting $m=n=0$ in (16.2) and allowing for (16.9) and (16.37), we get

$$U^{1/3}\left[8\pi\rho\,(t)-\frac{1}{12U^3}\left(\frac{dU}{dt}\right)^2\right]=\frac{1}{W_1}-\frac{1}{V_1W_1}\frac{d^2W_1}{dr^2}+\frac{1}{4V_1W_1^2}\left(\frac{dW_1}{dr}\right)^2+\frac{1}{2V_1^2W_1}\frac{dW_1}{dr}\frac{dV_1}{dr}.\tag{16.38}$$

Similarly, at $m=n=1$ (16.2) yields

$$U^{1/3}\left[8\pi p+\frac{1}{3U^2}\frac{d^2U}{dt^2}-\frac{5}{12U^3}\left(\frac{dU}{dt}\right)^2\right]=-\frac{1}{W_1}+\frac{1}{4V_1W_1^2}\left(\frac{dW_1}{dr}\right)^2.\tag{16.39}$$

Since the left-hand sides of (16.38) and (16.39) contain functions depending only on t and the right-hand sides contain functions depending only on r, we have

$$8\pi\rho\,(t)-\frac{1}{12U^3}\left(\frac{dU}{dt}\right)^2=c_0U^{-1/3},\tag{16.40}$$

$$\frac{1}{W_1}-\frac{1}{V_1W_1}\frac{d^2W_1}{dr^2}+\frac{1}{4V_1W_1^2}\left(\frac{dW_1}{dr}\right)^2+\frac{1}{2V_1^2W_1}\frac{dW_1}{dr}\frac{dV_1}{dr}=c_0,\tag{16.41}$$

$$8\pi p\,(t)+\frac{1}{3U^2}\frac{d^2U}{dt^2}-\frac{5}{12U^3}\left(\frac{dU}{dt}\right)^2=c_1U^{-1/3},\tag{16.42}$$

$$-\frac{1}{W_1}+\frac{1}{4V_1W_1^2}\left(\frac{dW_1}{dr}\right)^2=c_1,\tag{16.43}$$

where c_0 and c_1 are arbitrary constants.

Let us use (16.43) to find V_1:

$$V_1=\frac{1}{c_1W_1+1}\left(\frac{d\sqrt{W_1}}{dr}\right)^2.\tag{16.44}$$

Substitution of this expression into (16.41) reveals that the constants c_0 and c_1 are linked by the following relationship:

$$c_1=-\frac{1}{3}c_0.\tag{16.45}$$

Since $W_1(r)$ and $V_1(r)$ are expressed in terms of the same function $f(r)$ (see (16.25), (16.35), and (16.36)), (16.44) is actually an equation for $f(r)$. It will prove convenient, however, to take $W_1(r)$ for the independent function rather than $f(r)$ and to find the solution with respect to r as a function of $\sqrt{W_1}$.

Using (16.35) and (16.36), we can express $f(r)$ and $Z(r)$ in terms of $W_1(r)$ and $V_1(r)$ thus:

$$f(r) = W_1 \sqrt{V_1}, \tag{16.46}$$

$$Z(r) = (W_1^2 V_1^{-1})^{3/4}. \tag{16.47}$$

Finding the derivative of (16.47) with respect to r and allowing for (16.25) and (16.46), we get

$$r = \frac{\sqrt{W_1}}{V_1} \frac{d\sqrt{W_1}}{dr} - \frac{W_1}{4V_1^2} \frac{dV_1}{dr}.$$

Substituting (16.44) and performing certain manipulations, we will arrive at the sought equation for $r = r(\sqrt{W_1})$:

$$W_1(c_1 W_1 + 1) \frac{d^2 r}{(d\sqrt{W_1})^2} + \sqrt{W_1}(3c_1 W_1 + 2) \frac{dr}{d\sqrt{W_1}} - 2r = 0. \tag{16.48}$$

The metric tensor g_{ik} of the effective Riemann space-time is in RTG a function of the universal coordinates of the Minkowski space-time. This means that the Riemann space-time is specified on a single map and, therefore, in RTG there can be no closed homogeneous and isotropic universe, because such a universe is covered by at least two maps. Thus, the parameter c_1 in (16.44) is nonnegative.

Let us seek the solutions to Eq. (16.48) that are continuous, one-to-one, and unidirectional.

The general solution to Eq. (16.48) has the form

$$r(\sqrt{W_1}) = \frac{A_0}{3W_1}\left[\frac{\sqrt{W_1}}{2c_1}\sqrt{1 + c_1 W_1} - \frac{1}{2c_1^{3/2}} \ln\left(\sqrt{c_1 W_1} + \sqrt{1 + c_1 W_1}\right)\right] + \frac{B_0}{W_1}$$

$$\text{for } c_1 > 0 \tag{16.49}$$

and

$$r(\sqrt{W_1}) = \frac{A_1}{3}\sqrt{W_1} + \frac{B_1}{W_1} \quad \text{for} \quad c_1 = 0, \tag{16.50}$$

where solution (16.49) does not satisfy our requirements, and (16.50) satisfies them only for $B_1 = 0$.

Thus, the admissible solution to Eq. (16.48) is

$$r(\sqrt{W_1}) = \frac{A_1}{3}\sqrt{W_1}, \quad A_1 > 0. \tag{16.51}$$

Note that in the model of a universe considered here, where p and ρ depend only on time t, the fact that $c_1 = 0$ was established independently from Eqs. (16.40) and (16.42); hence, there is no way in which the presence of matter can affect the value of constant c_1. Since, in view of (16.45), $c_1 = 0$ implies $c_0 = 0$, if we allow for these values in Eqs. (16.40)-(16.43), we find that $U(t)$ is determined by the functions $\rho(t)$ and $p(t)$, while $V_1(r)$ and $W_1(r)$ are independent of these functions (in other words, complete separability is ensured).

Consequently, the solution just found, (16.51), is valid for empty space, too, and the constant A_1 is independent of the properties of matter. But since at $\rho = p = 0$ the line element (16.1) must coincide with the line element of the Minkowski space-time, $d\sigma^2 = dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$, we find that $A_1 = 3$. Thus, from (16.51) and (16.44) we finally obtain

$$W_1(r) = r^2, \quad V_1(r) = 1. \tag{16.52}$$

On the basis of (16.33), (16.34), and (16.52), we have the following formula for the line element (16.1) (see Appendix 5):

$$ds^2 = U(t)\,dt^2 - U^{1/3}(t)\,[dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\varphi^2)]. \tag{16.53}$$

If we now pass to proper time τ according to the formula

$$\sqrt{U(t)}\,dt = d\tau, \tag{16.54}$$

we can write the line element (16.53) as

$$ds^2 = d\tau^2 - U^{1/3}(\tau)\,[dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\varphi^2)]. \tag{16.55}$$

Comparing (16.55) with the well-known general formula for the Robertson-Walker metric,

$$ds^2 = d\tau^2 - U^{1/3}(\tau)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\,d\varphi^2)\right],$$

with constant k assuming values 0, +1, and −1, we conclude that in RTG the value of k is determined uniquely and is equal to zero.

Thus, in view of Eqs. (16.3), we inevitably arrive at the conclusion that in RTG the universe is infinite and flat.

If we introduce the notation

$$U^{1/3}(\tau) = R^2(\tau), \tag{16.56}$$

pass in (16.40) and (16.42), via (16.54), to proper time τ, and allow for the fact that $c_0 = c_1 = 0$, we obtain

$$\left(\frac{1}{R}\frac{dR}{d\tau}\right)^2 = \frac{8\pi}{3}\rho(\tau), \tag{16.57}$$

$$\frac{1}{R}\frac{d^2R}{d\tau^2} = -\frac{4\pi}{3}(3p+\rho). \tag{16.58}$$

Taking the derivative of both sides of (16.57) with respect to τ and allowing for (16.58), we arrive at the following expression for $R^{-1}\,dR/d\tau$:

$$\frac{1}{R}\frac{dR}{d\tau} = -\frac{1}{3(\rho+p)}\frac{d\rho}{d\tau}. \tag{16.59}$$

One can easily verify that this coincides with Eq. (16.18). Indeed, if in (16.18) we pass via (16.54) to proper time τ and allow for (16.56), we arrive at (16.59).

Let us now introduce the Hubble function

$$H(\tau) = \frac{1}{R}\frac{dR}{d\tau}. \tag{16.60}$$

For the present moment in the evolution of the universe, $\tau = \tau_0$, the value $H(\tau_0)$ is known as Hubble's constant and is positive. Hence, after we extract the root of (16.57) we must select the positive value

$$\frac{1}{R}\frac{dR}{d\tau} = \left(\frac{8\pi}{3}\rho\right)^{1/2}. \tag{16.61}$$

If for every finite τ the function $\rho(\tau)$ does not vanish, the Hubble function $H(\tau)$ is positive.

Equations (16.59) and (16.60) bring us to several general conclusions concerning the evolution in time of a homogeneous and isotropic universe.

Since $H(\tau)$ is positive, (16.60) implies that $dR/d\tau$ is positive, too. This means that $R(\tau)$ is a monotone increasing function of time τ, and because $\rho + p$ is

positive, (16.59) implies that $d\rho/d\tau$ is negative, with the result that $\rho(\tau)$ is a monotone decreasing function of τ.

If for every finite τ the function $\rho(\tau)$ does not vanish and the function $p(\tau)$ is nonnegative, then (16.58) implies $d^2R/d\tau^2 < 0$ and, since $dR/d\tau > 0$, the function $R = R(\tau)$ is monotone increasing and the respective curve is always convex upward. This means that over a finite time $\tau_{\min}$ in the past, $R(\tau)$ assumes its minimum value $R_{\min}(\tau_{\min}) = 0$. In what follows we assume $\tau_{\min}$ to be the reference point (the origin) for proper time τ and therefore can put $\tau_{\min} = 0$.

Let us define the critical density for every value of τ as follows:

$$\rho_c(\tau) = \frac{3}{8\pi} H^2(\tau). \tag{16.62}$$

Combining (16.57), (16.60), and (16.62), we get

$$\rho_c(\tau) \equiv \rho(\tau). \tag{16.63}$$

We see that the modern value of the density of matter in the universe, $\rho(\tau_0) = \rho_0$, must coincide with the critical value $\rho_c(\tau_0) = 10^{-29}$ g/cm^3 at the present moment in time. However, the observed density of matter in the universe, ρ_0, is almost 40 times lower than $\rho_c(\tau_0)$. Thus, RTG predicts the existence of a large "latent mass" in the universe in some form of matter, since this lacking mass is necessary for the identity $\rho_0 = \rho_c(\tau_0)$ to be valid.

Let us now study the system of equations (16.59), (16.61). The system is incomplete since there are only two equations in three unknowns $R(\tau)$, $\rho(\tau)$, and $p(\tau)$. Following Zel'dovich, 1961, for the third equation we take the equation of the state of matter in the simplest form,

$$p(\tau) = \nu\rho(\tau), \tag{16.64}$$

with ν being less than unity, since according to the hypothesis of Markov, 1983, the density of matter can never become infinite and, hence, the speed of sound must always be less than the speed of light.

Substituting (16.64) into (16.59) and integrating, we obtain

$$\rho(\tau) = \frac{\tilde{a}(\nu)}{R^{3(1+\nu)}}, \tag{16.65}$$

where $\tilde{a}(\nu)$ is the constant of integration with dimensions of g/cm^3. Combining (16.65) with Eq. (16.61), we get

$$R(\tau) = [6\pi\tilde{a}(\nu)(1+\nu)^2]^{1/3(1+\nu)}\tau^{2/3(1+\nu)}. \tag{16.66}$$

Note that at different stages in the evolution of the universe the parameter ν and, hence, $\tilde{a}(\nu)$ have different values. For example, in the radiation-dominated era, $\nu = 1/3$, and in this case

$$\rho(\tau) = aR^{-4}(\tau), \tag{16.67}$$

where $a = \tilde{a}(1/3)$, and

$$R(\tau) = \left(\frac{32a\pi}{3}\right)^{1/4} \tau^{1/2}. \tag{16.68}$$

In the nonrelativistic era, when we can neglect the pressure in comparison to the density of matter, in (16.64)-(16.66) we must set ν equal to zero. This yields

$$\rho(\tau) = bR^{-3}(\tau), \tag{16.69}$$

with $b = \tilde{a}(0)$, and

$$R(\tau) = (6\pi b)^{1/3}\,\tau^{2/3}. \tag{16.70}$$

The present stage in the evolution of the universe belongs to this era.

The equation of state of matter (16.64) with $1/3 < \nu < 1$ may exist (if it is realized at all) only at the very early stage in the evolution of the universe, even before the radiation-dominated era, and at a limiting value of the density of matter equal to

$$\rho_{\text{Pl}} = 5.157 \times 10^{93}\ \text{g/cm}^3. \tag{16.71}$$

Note that the equation of state with the limiting value of parameter ν equal to unity has been considered in Belinskii and Khalatnikov, 1972, and Belinskii *et al.*, 1985.

A parameter often introduced for cosmological measurements is the dilation parameter

$$q(\tau) = -R\frac{d^2R}{d\tau^2}\Big/\left(\frac{dR}{d\tau}\right)^2. \tag{16.72}$$

By using (16.70) we can easily show that at the present stage in the evolution of the universe,

$$q = 1/2. \tag{16.73}$$

Another important corollary of RTG can be obtained as follows. If we combine (16.33), (16.34), and (16.52) with (8.28), it is easy to show that $t^{00} = 0$, that is, the total energy density of matter and gravitational field for a Friedmann universe in the Minkowski space-time is zero.

In Chapter 11 we set up a pair of equations, (11.9) and (11.10), that describe a massive gravitational field. Let us study the model of a homogeneous and isotropic universe from the standpoint of these equations (see Appendix 5).

Since the covariant conservation law $\nabla_m T^m_n = 0$ and Eq. (11.10) are also valid for a massive gravitational field, all formulas derived from these equations are valid in this case. We wish to demonstrate that here, that is, when the graviton mass m is nonzero, the energy density t^{00} vanishes. Indeed, since $\sqrt{-g} = U(t)\, r^2 \sin\theta$ for the line element (16.53) and, hence, $\tilde{g}^{00} = r^2 \sin\theta$, and if we allow for the fact that $\tilde{\gamma}^{00} = r^2 \sin\theta$ in spherical coordinates, then (8.1) yields $\tilde{\Phi}^{00} = 0$. This, together with (11.14), forces us to conclude that for $m \neq 0$, for a Friedmann universe, the energy density t^{00} of matter and gravitational field taken together is always zero.

We will allow for the graviton mass in the following manner. We will take (16.53) as the expression for the line element in the Riemann space-time and use Eq. (11.9) to find $U(t)$, $\rho(t)$, and $p(t)$. Up to this point, in order to simplify the formulas we used the system of units in which $c = \hbar = G = 1$, but for the case at hand we turn to the centimeter-gram-second system.

In what follows it is convenient to refer to Eqs. (11.9) as written in terms of mixed components:

$$\sqrt{-g}\left(R^m_n - \frac{1}{2}\delta^m_n R\right) + \frac{\sqrt{-g}}{2}\left(\frac{mc}{\hbar}\right)^2\left[\delta^m_n + g^{mk}\gamma_{kn} - \frac{1}{2}\delta^m_n g^{pk}\gamma_{pk}\right] = \varkappa T^m_n, \tag{16.74}$$

where we have introduced the notation $\varkappa = 8\pi G/c^2$.

If in (16.54) we pass to proper time and allow for notation (16.56), then combining (16.74) with (16.9) yields

$$\left(\frac{1}{R}\frac{dR}{d\tau}\right)^2=\frac{c^2}{3}\varkappa\rho(\tau)-\frac{\varepsilon}{R^6}(R^2-1)^2\left(R^2+\frac{1}{2}\right), \tag{16.75}$$

$$\frac{1}{R}\frac{d^2R}{d\tau^2}=-\frac{c^2}{6}\varkappa\rho(\tau)-\frac{1}{2}\varkappa p(\tau)-\varepsilon\left(1-\frac{1}{R^6}\right), \tag{16.76}$$

with

$$\varepsilon=\frac{1}{6}\left(\frac{mc^2}{\hbar}\right)^2. \tag{16.77}$$

Finding the derivative of both sides of (16.75) with respect to τ and allowing for (16.75) and (16.76), we arrive at the following expression for $R^{-1}\,dR/d\tau$:

$$\frac{1}{R}\frac{dR}{d\tau}=-\frac{1}{3(\rho+p/c^2)}\frac{d\rho}{d\tau}, \tag{16.78}$$

which coincides with (16.59).

A qualitative analysis of Eqs. (16.75) and (16.76) enables drawing a number of important conclusions concerning the evolution of the universe. To this end we will consider the right-hand side of Eq. (16.75), but first let us study the behavior of the function

$$f(R)=\frac{\varepsilon}{R^6}(R^2-1)^2\left(R^2+\frac{1}{2}\right).$$

If we send R to zero,

$$f(R)\simeq\frac{\varepsilon}{2R^6}, \tag{16.79}$$

while if we send R to infinity,

$$f(R)\simeq\varepsilon. \tag{16.80}$$

Also, $f(1)=0$ at $R=1$. Since

$$\frac{df}{dR}=\frac{3\varepsilon}{R^7}(R^2-1)(R^2+1),$$

the function $f(R)$ monotonically decreases for $R\in[0,\,1]$, assuming a minimal value equal to zero at $R=1$, or $f(1)=0$, and for $R>1$ monotonically increases, asymptotically approaching the value (16.80).

In view of (16.65), which is a corollary of (16.59), and the equation of state (16.64), which in the centimeter-gram-second system has the form

$$p(\tau)=\nu c^2\rho(\tau), \tag{16.81}$$

we conclude that the right-hand side of (16.75) is positive if $R(\tau)$ satisfies the following inequalities:

$$R_{\min}\leqslant R(\tau)\leqslant R_{\max}, \tag{16.82}$$

where $R_{\min}$ is positive and $R_{\max}$ finite. Here the nonzero lower bound on $R(\tau)$ occurs because $\nu<1$, while the upper bound always exists because the density of matter $\rho(\tau)$, according to (16.65), monotonically decreases, as R increases, and tends to zero, while $f(R)$ tends to ε as $R\to\infty$. This means that a Friedmann universe performs cyclic evolution, with the density of matter remaining always finite.

According to Eq. (16.75), in the range from $R_{\min}$ to $R_{\max}$ we have

$$\frac{1}{R}\frac{dR}{d\tau}=\left[\frac{c^2\varkappa}{3}\rho(\tau)-\frac{\varepsilon}{R^6}(R^2-1)^2\left(R^2+\frac{1}{2}\right)\right]^{1/2}, \tag{16.83}$$

while in the range from $R_{\max}$ to $R_{\min}$ we have

$$\frac{1}{R}\frac{dR}{d\tau}=-\left[\frac{c^2\varkappa}{3}\rho(\tau)-\frac{\varepsilon}{R^6}(R^2-1)^2\left(R^2+\frac{1}{2}\right)\right]^{1/2}. \qquad (16.84)$$

Let us establish the behavior of the R vs. τ curve at points $R=R_{\max}$ and $R=R_{\min}$.

If $p(\tau)\geqslant 0$, for $R\geqslant 1$ the right-hand side of Eq. (16.76) is always negative; hence $d^2R/d\tau^2<0$, which means that the function $R=R(\tau)$ assumes its maximum value at the point where $dR/d\tau=0$.

Let us determine the sign of $d^2R/d\tau^2$ at point $R=R_{\min}<1$. If we allow for the equation of state, (16.81), taken at point $\tau=0$,* from (16.76) we get

$$\left(\frac{1}{R}\frac{d^2R}{d\tau^2}\right)_{\tau=0}=-\frac{c^2\varkappa}{3}\rho(0)\frac{1+3\nu}{2}-\varepsilon\left(1-\frac{1}{R_{\min}^6}\right). \qquad (16.85)$$

But since $\rho(0)$ and $R_{\min}$ are linked by the relationship

$$\frac{c^2\varkappa}{3}\rho(0)-\frac{\varepsilon}{R_{\min}^6}(R_{\min}^2-1)^2\left(R_{\min}^2+\frac{1}{2}\right)=0,$$

allowing for it in (16.85) yields

$$\left(\frac{1}{R}\frac{d^2R}{d\tau^2}\right)_{\tau=0}=\varepsilon\left[\frac{3(1+3\nu)}{4R_{\min}^2}+\frac{3(1-\nu)}{4R_{\min}^6}-\frac{3(1+\nu)}{2}\right].$$

Since $R_{\min}<1$, we obtain

$$\left(\frac{d^2R}{d\tau^2}\right)_{\tau=0}>0.$$

Hence, the function $R=R(\tau)$ assumes its minimum at $\tau=0$.

Let us now examine the behavior of the Hubble function (16.60) for $R\in(R_{\min}, R_{\max})$. Allowing in (16.76) for the equation of state (16.81), we obtain

$$\frac{1}{R}\frac{d^2R}{d\tau^2}=-\frac{c^2\varkappa}{3}\rho(\tau)\frac{1+3\nu}{2}-\varepsilon\left(1-\frac{1}{R^6}\right). \qquad (16.86)$$

Finding $(1/3)\,c^2\varkappa\rho(\tau)$ from (16.75), substituting its value into (16.86), and performing relatively simple manipulations, with allowance made for the definition (16.60) of the Hubble function, we find that

$$\frac{dH}{d\tau}+\frac{3(1+\nu)}{2}H^2=\varepsilon\Psi(R), \qquad (16.87)$$

where

$$\Psi(R)=\frac{3(1+3\nu)}{4R^2}+\frac{3(1-\nu)}{4R^6}-\frac{3(1+\nu)}{2}. \qquad (16.88)$$

It is easily noted that

$$\Psi(R)\text{ is }\begin{cases}\text{positive if } R<1,\\ \text{zero if } R=1,\\ \text{negative if } R>1.\end{cases} \qquad (16.89)$$

Since $H(\tau)$ vanishes only at points $R(\tau)=R_{\min}$ and $R(\tau)=R_{\max}$, while $H(\tau)\neq 0$ for other values of R belonging to the interval $(R_{\min}, R_{\max})$, Eq. (16.87) can be represented in the form

$$-\frac{d}{d\tau}\left(\frac{1}{H}\right)+\frac{3(1+\nu)}{2}=\frac{\varepsilon}{H^2}\Psi(R). \qquad (16.90)$$

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* With $\tau=0$ we associate the value $R(0)=R_{\min}$.

Since $\Psi(R)$ is negative for $1 < R < R_{\max}$, (16.90) yields the inequality $dH^{-1}/d\tau > 0$; hence, the Hubble function $H(\tau)$ monotonically decreases for $R \in (1, R_{\max})$.

From (16.90) it readily follows that $H(\tau)$ monotonically decreases for $R \in (R_s, 1]$, too, with R_s the solution to the equation

$$\frac{3}{2}(1+\nu)H^2 = \varepsilon\Psi(R), \tag{16.91}$$

and monotonically increases for $R \in (R_{\min}, R_s)$, since in this interval $dH^{-1}/d\tau < 0$.

Hence, the Hubble function $H(\tau)$ assumes its maximum at $R = R_s$. Let us find the value of R_s. If we combine Eq. (16.75) with (16.65) and (16.88), then (16.91) yields

$$\frac{1+\nu}{2}\widetilde{A}(\nu) = \left(\frac{1}{R_s^4} - 1\right)R_s^{1+3\nu}, \tag{16.92}$$

where for the sake of convenience we introduce the notation

$$\widetilde{A}(\nu) = \frac{32\pi G\hbar^2}{c^4}\left(\frac{\widetilde{a}(\nu)}{m^2}\right). \tag{16.93}$$

Since $R_s \ll 1$, Eq. (16.92) yields

$$R_s \simeq \left[\frac{1}{2}(1+\nu)\widetilde{A}(\nu)\right]^{-1/3(1-\nu)}. \tag{16.94}$$

Combining (16.83) with the expression (16.65) for $\rho(\tau)$ and substituting (16.94) into the combination yields

$$H_{\max} \simeq \sqrt{\varepsilon}\left(\frac{1-\nu}{1+\nu}\right)^{1/2}\left(\frac{1+\nu}{2}\widetilde{A}(\nu)\right)^{1/(1-\nu)}. \tag{16.95}$$

To establish the nature of the time evolution of a Friedmann universe we must solve Eqs. (16.83). It is convenient to seek the solution in the form $\tau = \tau(R)$:

$$\tau = \int\limits_{R_{\min}}^{R} \frac{dx}{x}\left[\frac{1}{3}c^2\varkappa\rho(x) - \frac{\varepsilon}{x^6}(x^2-1)^2(x^2+1/2)\right]^{-1/2}. \tag{16.96}$$

In the early stages of the evolution of the universe, when the value of ν lies between 1/3 and 1, solution (16.96) can be written as

$$\tau = \sqrt{\frac{2}{\varepsilon}}\int\limits_{R_{\min}}^{R} \frac{x^2\,dx}{[\widetilde{A}x^{3(1-\nu)} - 2x^6 + 3x^4 - 1]^{1/2}}, \tag{16.97}$$

where $\widetilde{A}$ is given by (16.93) and constitutes a large number, since, as will shortly be demonstrated, the graviton mass m is extremely small.

In deriving (16.97) we allowed for (16.65). As noted earlier, states with ν ranging from 1/3 to 1 in Eq. (16.81) may be realized at very early stages of the evolution of the universe. It is therefore natural to assume that

$$R(\tau) \ll (\widetilde{A}/2)^{1/3(1+\nu)}. \tag{16.98}$$

In view of this, the integral (16.97), which determines the τ vs. R dependence, can be approximately replaced with

$$\tau = \sqrt{\frac{2}{\varepsilon}}\int\limits_{R_{\min}}^{R} \frac{x^2\,dx}{[\widetilde{A}x^{3(1-\nu)} - 1]^{1/2}}. \tag{16.99}$$

For $\nu \neq \nu_n = (n - 1/2)/(n + 1/2)$, with n an integer greater than unity, (16.99) yields

$$\tau = \frac{2}{3}\left(\frac{2}{\varepsilon}\right)^{1/2} \frac{1}{1-\nu}\, \widetilde{A}^{-1/(1-\nu)} u^{1/2} F\left(-\frac{\nu}{1-\nu}; \frac{1}{2}; \frac{3}{2}; -u\right), \qquad (16.100)$$

where $F(a; b; c; z)$ is the hypergeometric function,

$$u = \left(\frac{R}{R_{\min}}\right)^{3(1-\nu)} - 1, \qquad (16.101)$$

and

$$R_{\min} \simeq \widetilde{A}^{-1/3(1-\nu)}. \qquad (16.102)$$

For $\nu = \nu_n$ (16.99) yields

$$\tau = \left(\frac{2}{\varepsilon}\right)^{1/2} \frac{2n+1}{3}\, \widetilde{A}^{-(n+1/2)} \left\{ \frac{u^{1/2}(1+u)^{1/2}}{2n} \left[(1+u)^{n-1} + \sum_{k=1}^{n-k} \frac{1}{2^k} \frac{n!\,(2n-1)!!\,(1+u)^{n-k-1}}{(n-k-1)!\,(2n-2k-1)!!} \right] + \frac{(2n-1)!!}{2^n n!} \ln\left[u + (1+u)^{1/2}\right] \right\}. \qquad (16.103)$$

Let us first investigate the asymptotic behavior of $\tau(R)$ in the region where $R \gg R_{\min}$, that is, where $u \gg 1$. Since in this case (see Abramowitz and Stegun, 1964, and Erdélyi, 1955)

$$F\left(-\frac{\nu}{1-\nu}; \frac{1}{2}; \frac{3}{2}; -u\right) \simeq \frac{1-\nu}{1+\nu}\, u^{\nu/(1-\nu)}, \qquad (16.104)$$

(16.100) yields

$$\tau \simeq \frac{2}{3}\left(\frac{2}{\varepsilon}\right)^{1/2} \frac{1}{1+\nu}\, \widetilde{A}^{-1/(1-\nu)} u^{(1/2)(1+\nu)/(1-\nu)}. \qquad (16.105)$$

Combining this with (16.101) yields

$$R(\tau) \simeq \left[\frac{9\varepsilon\widetilde{A}(1+\nu)^2}{8}\right]^{1/3(1+\nu)} \tau^{2/3(1+\nu)}, \quad \nu \neq \nu_n. \qquad (16.106)$$

Since $\varepsilon\widetilde{A} = 2\widetilde{a}c^2\varkappa/3$, the leading asymptotic term (16.106) in the expansion of $R(\tau)$ is independent of the graviton mass. For $\nu = \nu_n$ and $u \gg 1$, (16.103) yields

$$R(\tau) \simeq \left[\frac{9n^2\varepsilon\widetilde{A}}{2\left(n+\frac{1}{2}\right)^2}\right]^{(2n+1)/12n} \tau^{(2n+1)/6n}. \qquad (16.107)$$

It is readily noted that (16.107) coincides with (16.106) if in the latter we formally put $\nu = \nu_n$. Obviously, (16.107) is also independent of the graviton mass.

We now wish to examine the asymptotic region where $R(\tau) - R_{\min} \ll R_{\min}$. In this case (16.100) and (16.103) yield

$$\tau = \frac{2}{3}\left(\frac{2u}{\varepsilon}\right)^{1/2} \times \begin{cases} \dfrac{1}{1-\nu}\, \widetilde{A}^{-1/(1-\nu)} & \text{if } \nu \neq \nu_n, \\ C_n \widetilde{A}^{n+1/2} & \text{if } \nu = \nu_n, \end{cases} \qquad (16.108)$$

where by C_n we denote a numerical coefficient,

$$C_n = \frac{2n+1}{4n}\left[1 + \frac{3(2n-1)!!}{2^n(n-1)!} + n!\,(2n-1)!! \sum_{k=1}^{n-1} \frac{1}{(n-k-1)!\,(2n-2k-1)!!}\right]. \qquad (16.109)$$

Solving (16.108) for R, we obtain

$$R(\tau) \simeq \widetilde{A}^{-1/3(1-\nu)} + \frac{3\varepsilon}{8}\widetilde{A}^{5/3(1-\nu)}(1-\nu)\tau^2 \text{ if } \nu \neq \nu_n, \tag{16.110}$$

$$R(\tau) \simeq \widetilde{A}^{-(2n+1)/6} + \frac{3\varepsilon}{8}\widetilde{A}^{5(2n+1)/6}\frac{2n+1}{2C_n^2}\tau^2 \text{ if } \nu = \nu_n. \tag{16.111}$$

Note that these formulas are valid for low values of τ:

$$\tau \ll \widetilde{A}^{-1/(1-\nu)}\frac{\varepsilon^{-1/2}}{(1-\nu)^{1/2}}. \tag{16.112}$$

Interestingly, for every fixed value of ν varying between 1/3 and 1 all the asymptotic relations (16.110) and (16.111) are represented by parabolas, which follows directly from the positivity of the second derivative $d^2R/d\tau^2$ at the minimum point $R = R_{\min}$.

The Hubble function $H(\tau)$ in this asymptotic region is represented thus:

$$H(\tau) \simeq \begin{cases} \dfrac{3\varepsilon}{4}\widetilde{A}^{2/(1-\nu)}(1-\nu)\tau \text{ if } \nu \neq \nu_n, \\ \dfrac{3\varepsilon}{4}\widetilde{A}^{2n+1}\dfrac{2n+1}{C_n^2}\tau \text{ if } \nu = \nu_n. \end{cases} \tag{16.113}$$

The next step in the evolution of the universe on the ν scale corresponds to $\nu = 1/3$. In this case the integral (16.97) can be represented thus:

$$\tau = \tau_{1/3} + \frac{1}{2\sqrt{\varepsilon}}\int_{R_{1/3}^2}^{R^2}\left(\frac{y}{(y-y_3)(y-y_2)(y_1-y)}\right)^{1/2}dy, \tag{16.114}$$

where $\tau_{1/3}$ is the time that it took the universe to evolve from the beginning of the expansion stage right up to the beginning of the ultrarelativistic (radiation-dominated) state of matter (the value $\nu = 1/3$ in Eq. (16.81)), $R_{1/3} = R(\tau_{1/3})$, and y_1, y_2, and y_3 are the roots of the cubic equation $Ay - 2y^3 + 3y^2 - 1 = 0$, whose approximate values are

$$\begin{aligned} y_1 &= \left(\frac{A}{2}\right)^{1/2} + \frac{3}{4} + \frac{9}{(32A)^{1/2}} + O\left(\frac{1}{A}\right), \\ y_2 &= -\left(\frac{A}{2}\right)^{1/2} + \frac{3}{4} - \frac{9}{(32A)^{1/2}} + O\left(\frac{1}{A}\right), \\ y_3 &= \frac{1}{A} + O\left(\frac{1}{A^3}\right), \end{aligned} \tag{16.115}$$

with $A = \widetilde{A}(1/3)$. Since $y \ll (A/2)^{1/2}$, we have $|y/y_1| \ll 1$ and $|y/y_2| \ll 1$, and therefore (16.114) may be replaced with

$$\tau = \tau_{1/3} + \frac{1}{\sqrt{2\varepsilon A}}\int_{R_{1/3}^2}^{R^2}\left(\frac{y}{y-y_3}\right)^{1/2}dy. \tag{16.116}$$

After evaluating the integral and allowing for the fact that $y_3 \simeq 1/A$ we obtain

$$\tau = \tau_{1/3} + \frac{1}{\sqrt{2\varepsilon A}}\left[R\sqrt{R^2 - A^{-1}} - R_{1/3}\sqrt{R_{1/3}^2 - A^{-1}} + \frac{1}{A}\ln\frac{R + \sqrt{R^2 - A^{-1}}}{R_{1/3} + \sqrt{R_{1/3}^2 - A^{-1}}}\right]. \tag{16.117}$$

Since the radiation-dominated era in the evolution of the universe ($\nu = 1/3$) sets in after the stage where the state of matter in the universe is described by Eq. (16.81) with $\nu > 1/3$, it is only natural to assume that $R_{1/3} \gg A^{-1}$.

Let us discuss the limiting modes of (16.117). First let $R \gg R_{1/3}$. Then (16.117) implies

$$R(\tau) \simeq (2\varepsilon A)^{1/4}(\tau - \tau_{1/3})^{1/2}. \tag{16.118}$$

If we combine this with the notations (16.77) and (16.93) and the inequality $\tau \gg \tau_{1/3}$, we get

$$R(\tau) \simeq \left(\frac{32\pi aG}{3}\right)^{1/4} \tau^{1/2}, \tag{16.119}$$

which coincides with (16.68)* established earlier for the case where $m = 0$.

Now suppose that $R - R_{1/3} \ll R_{1/3}$. Then (16.117) implies

$$R^2 \simeq R_{1/3}^2 + (2\varepsilon A)^{1/2}(\tau - \tau_{1/3}). \tag{16.120}$$

It cannot be ruled out that the entire early universe was totally in the ultrarelativistic state. This asks for a special investigation. The respective formulas can easily be obtained from (16.116) if we put $\tau_{1/3} = 0$ and $R_{1/3} = R_{\min} = A^{-1/2}$. Then the expression (16.117) for τ assumes the form

$$\tau = \frac{1}{\sqrt{2\varepsilon A}}\left[R\sqrt{R^2 - A^{-1}} + A^{-1}\ln\left(\sqrt{A}\,R + \sqrt{AR^2 - 1}\right)\right]. \tag{16.121}$$

Naturally, for $R \gg R_{\min}$ formula (16.121) is transformed into (16.119), while for $R - R_{\min} \ll R_{\min}$ formula (16.121) yields

$$R(\tau) \simeq A^{-1/2} + \varepsilon A^{5/2}\tau^2. \tag{16.122}$$

On the basis of this it is easy to find the explicit form of the Hubble function $H(\tau)$:

$$H(\tau) = 2\varepsilon A^3 \tau. \tag{16.123}$$

Note that (16.122) and (16.123) are valid for times $\tau \ll \varepsilon^{-1/2}A^{-3/2}$.

Let us now turn to the nonrelativistic stage in the evolution of the universe, when the pressure is much lower than the density of matter and can be ignored ($\nu = 0$). The integral (16.96) can then be written thus:

$$\tau = \tau_{\text{nonrel}} + \sqrt{\frac{2}{\varepsilon}} \int\limits_{R_{\text{nonrel}}}^{R} \frac{x^2\,dx}{(Bx^3 - 2x^6 + 3x^4 - 1)^{1/2}}, \tag{16.124}$$

where τ_{nonrel} is the time it took the universe to evolve from the beginning of the expansion stage right up to the beginning of the nonrelativistic stage, $R_{\text{nonrel}} = R(\tau_{\text{nonrel}})$, and

$$B = \frac{32\pi G\hbar^3}{c^4}\frac{b}{m^2}, \tag{16.125}$$

$b = \tilde{a}(0)$. In deriving (16.124) we allowed for formula (16.65) for $\rho(\tau)$ and since b is a relatively large constant, B is a large parameter as well.

Let us introduce the notation

$$R_0 = [(1/2 - \alpha)B]^{1/3}, \tag{16.126}$$

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* Provided that in (16.68) we reintroduce the dependence on G.

where α is a small positive number. It can easily be verified that for all values of $R(\tau)$ belonging to the interval $[R_{\text{nonrel}}, R_0]$ we have $BR^3 - 2R^6 \gg 3R^4 - 1$, provided that we have selected α such that

$$\alpha \gg B^{-2/3}. \tag{16.127}$$

If in the denominator of the integrand in (16.124) we discard $3x^4 - 1$ and integrate, we get

$$\tau - \tau_{\text{nonrel}} = \frac{2}{3\sqrt{\varepsilon}}\left\{\sin^{-1}\left[\left(\frac{2}{B}\right)^{1/2} R^{3/2}\right] - \sin^{-1}\left[\left(\frac{2}{B}\right)^{1/2} R^{3/2}_{\text{nonrel}}\right]\right\}. \tag{16.128}$$

For times $\tau \gg \tau_{\text{nonrel}}$ and $R_{\text{nonrel}} \ll B^{1/3}$ this yields

$$R(\tau) \simeq \left(\frac{B}{2}\right)^{1/3} \sin^{2/3}\left(\frac{3}{2}\sqrt{\varepsilon}\,\tau\right). \tag{16.129}$$

Let us estimate the maximum value of τ for which (16.129) still remains valid. Since $R(\tau)$ may reach the value $R_0 = [(1/2 - \alpha) B]^{1/3}$, from (16.129) we get

$$\tau_0 \simeq 2\left(\frac{2}{3}\right)^{1/2} \frac{\hbar}{mc^2} \sin^{-1}(1-\alpha). \tag{16.130}$$

For times $\tau > \tau_0$ we write (16.124) in the form

$$\tau = \tau_0 + \sqrt{\frac{2}{\varepsilon}} \int_{R_0}^{R} \frac{x^2\,dx}{(Bx^3 - 2x^6 + 3x^4 - 1)^{1/2}}. \tag{16.131}$$

We introduce the notation

$$R_1 = \left(\frac{B}{2}\right)^{1/3} + \left(\frac{1}{4B}\right)^{1/3}. \tag{16.132}$$

It can easily be demonstrated that for all values of $R(\tau)$ belonging to the interval $[R_0, R_1]$ we can discard the 1 in the denominator of the integrand in (16.131) and write

$$\tau = \tau_0 + \sqrt{\frac{2}{\varepsilon}} \int_{R_0}^{R} \frac{x^{1/2}\,dx}{(B - 2x^3 + 3x)^{1/2}}. \tag{16.133}$$

The approximate values of the roots of the equation $B - 2x^3 + 3x = 0$ are

$$x_1 \simeq R_1, \quad x_{2,3} \simeq -\frac{R_1}{2} \pm \frac{i\sqrt{3}}{2} R_2, \tag{16.134}$$

with

$$R_2 = \left(\frac{B}{2}\right)^{1/3} - \left(\frac{1}{4B}\right)^{1/3}. \tag{16.135}$$

We can therefore write (16.133) as follows:

$$\tau = \tau_0 + \frac{1}{\sqrt{2\varepsilon}} \int_{R_0}^{R} \frac{dx}{[x(R_1 - x)]^{1/2}} \left[\frac{x^2}{(x + R_1/2)^2 + 3(R_2/2)^2}\right]^{1/2}.$$

Applying the mean-value theorem, we obtain

$$\tau = \tau_0 + \frac{\mu_0}{\sqrt{2\varepsilon}} \int_{R_0}^{R} \frac{dx}{[x(R_1 - x)]^{1/2}}, \tag{16.136}$$

with

$$\mu_0 = \left[\left(1+\frac{R_1}{2\xi}\right)^2 + 3\left(\frac{R_2}{2\xi}\right)^2\right]^{-1/2}, \tag{16.137}$$

where ξ is a number belonging to the interval $[R_0, R]$. Although, generally speaking, the value of μ_0 depends on R, this dependence is extremely weak, since for all the values of R considered here the value of μ_0 is limited by the following inequalities:

$$\frac{1}{\sqrt{3}}\left(1-\frac{\alpha}{3}\right) < \mu_0 < \frac{1}{\sqrt{3}}. \tag{16.138}$$

Integration in (16.136) yields

$$\tau = \tau_0 + \frac{\mu_0}{\sqrt{2\varepsilon}}\left[\sin^{-1}\left(\frac{R_1-2R}{R_1}\right) - \sin^{-1}\left(\frac{R_1-2R_0}{R_1}\right)\right]. \tag{16.139}$$

This enables calculating the approximate value of the expansion period of the universe. Assuming that $R = R_1$ in (16.139) and taking into account (16.126) and (16.132), we find that

$$\tau_1 \simeq \tau_0 + \frac{2\mu_0}{\sqrt{\varepsilon}}\left(\frac{\alpha}{3}\right)^{1/2}. \tag{16.140}$$

Substituting (16.130) and allowing for (16.138) yields

$$\left(\frac{2}{3}\right)^{1/2}\left[\pi - \frac{(2\alpha)^{3/2}}{3}\right]\frac{\hbar}{mc^2} < \tau_1 < \left(\frac{2}{3}\right)^{1/2}\frac{\pi\hbar}{mc^2}, \tag{16.141}$$

where α is a sufficiently small positive number.

We note that the half-period of the cyclic evolution of the universe, τ_1, is determined by the value of the graviton mass, in addition to the values of the fundamental constants $\hbar$ and c.

Let us now establish the upper bound for the graviton mass. Basing our reasoning on the condition that the age of the universe, which is determined via Hubble's constant $H(\tau_0)$, must be smaller than the cyclic half-period,

$$\frac{2}{3H(\tau_0)} < \left(\frac{2}{3}\right)^{1/2}\frac{\pi\hbar}{mc^2},$$

we arrive at the following bound on the graviton mass:

$$m < \left(\frac{3}{2}\right)^{1/2}\frac{\pi\hbar H(\tau_0)}{c^2}. \tag{16.142}$$

Experimentally Hubble's constant $H(\tau_0)$ has yet to be determined more exactly. The current value of the constant lies between the following two values:

$$(55\pm 5)\ (\text{km/s})/\text{Mpc} \leqslant H(\tau_0) \leqslant (110\pm 10)\ (\text{km/s})/\text{Mpc}.$$

This combined with (16.142) provides the following upper bound for the graviton mass:

$$m \leqslant 10^{-66}\text{-}10^{-65}\ \text{g}. \tag{16.143}$$

Note that the fact that the graviton has a finite mass increases the deficit of the "latent mass" of the universe if compared to the case of a massless graviton. Indeed, according to (16.75), for the present moment in the evolution of the universe $(R \gg 1)$ we have

$$\rho = \rho_c + \frac{1}{2\varkappa}\left(\frac{mc}{\hbar}\right)^2, \tag{16.144}$$

where, by definition, the critical density is

$$\rho_c(\tau) = \frac{3H^2}{8\pi G}. \tag{16.145}$$

At $m = 10^{-65}$ g the expression for the effective gravitational density $\rho_g = \frac{1}{2\varkappa}\left(\frac{mc}{\hbar}\right)^2$ yields $\rho_g = 0.72 \times 10^{-29}$ g/cm^3. When, say, $H(\tau_0) \simeq 75$ (km/s)/Mpc, the value of ρ_c proves to be 1.04×10^{-29} g/cm^3, and the "latent mass" deficit increases twofold against the value for the case of a massless graviton, which means that the "latent mass" deficit amounts to 80 times the observable mass.

Note that if we assume that the total rest mass of the three types of neutrino is 110 eV, the origin of the mass deficit can be explained.

Let us now discuss the value of the experimentally measurable dilation parameter (16.72). For $R \gg 1$ this parameter is

$$q(\tau) \simeq \frac{1}{2}\left(1 + \frac{3\rho_g}{\rho_c(\tau)}\right). \tag{16.146}$$

This demonstrates that ρ_g and, hence, the graviton mass can be expressed in terms of measurable quantities, the dilation parameter q and the Hubble function H. As $R \to R_{\max}$, the dilation parameter q tends to $+\infty$, since $\rho_c(\tau)$ tends to zero.

To conclude this chapter we note that Olber's paradox is absent from the given scheme, since in the contraction cycle at a temperature $T = 4 \times 10^3$ K ($\rho \simeq 10^{-20}$ g/cm^3) hydrogen ionization sets in and the universe becomes opaque, which ensures that the integral luminosity of the stars has a finite value.

Thus, introduction of a graviton with a finite mass drastically changes the nature of the evolution of a homogeneous and isotropic universe (Friedmann's universe). Such a universe exists for an infinitely long time, oscillates, and which is most important, the density of matter in such a universe will always be finite. If we take this universe at a certain time $\tau = 0$ for which $R(0) = R_{\min}$, then over a time interval equal to τ_1 it expands to a value $R(\tau_1) = R_{\max}$, then contracts, and $R(\tau)$ will return to $R_{\min}$ over a time interval τ_1. The process is repeated infinitely.

Chapter 17. Post-Newtonian Approximation in RTG

The post-Newtonian approximation was set up to study systems of the island type moving with low, or nonrelativistic, velocities. Hence, to construct perturbation series necessary for problems of this kind it is natural to select the ratio $v/c = \varepsilon$ as the small parameter. Since any correct theory of gravitation must lead us to Newton's law of universal gravitation in the limit of small relative velocities of the objects involved, and since in Newton's theory

$$\left(\frac{v}{c}\right)^2 = \frac{GM}{rc^2} \tag{17.1}$$

exactly, the post-Newtonian approximation may be used in the study of motion involving relatively small heavenly bodies. As experimental data show, the value of the dimensionless Newtonian potential GM/rc^2, say, on the surface of the Sun (and, hence, on the surfaces of heavenly bodies of the Sun type), does not exceed 2×10^{-6}, and for the Earth this quantity at the surface is 6.95×10^{-9}. It is also known that in the solar system the specific pressure $p/c^2\rho$ and the specific internal energy Π are of roughly the same order of smallness, $\varepsilon^2 \sim 10^{-6}$. This

means that within the solar system the value of ε can be used as the small parameter in the perturbation series expansions of the post-Newtonian type. And the first few terms in the series expansions may be expected to describe sufficiently well the entire range of phenomena occurring in the solar system.

A characteristic feature of the solar system is the fact that in the system of units in which $c = 1$ the velocities of the various movements do not exceed ε, which means that in order of magnitude the spatial and temporal partial derivatives are linked by the following relationship:

$$\frac{\partial}{\partial t} \sim \varepsilon \frac{\partial}{\partial x^\alpha}. \tag{17.2}$$

This formula implies that all time variations of the quantities are primarily associated with the motion of matter.

Since for the solar system such characteristics as the dimensionless Newtonian potential GM/rc^2, the specific pressure, the specific internal energy, and $(v/c)^2$, with v the velocity of orbital motion of the planets, are of the order of $\varepsilon^2 \sim 10^{-6}$, the aim of the post-Newtonian approximation is to find the correction terms in the next order in ε.

In this chapter we will build the post-Newtonian approximation for the RTG system of equations (8.36), (8.37). Even if the graviton rest mass is finite, in view of its extreme smallness it can play no important role within the solar system. Hence, it becomes sufficient to study only Eqs. (8.36) and (8.37). To simplify matters, in what follows we employ a system of units in which $c = 1$.

We start with the expansions

$$g_{00} = 1 + \overset{(2)}{g}_{00} + \overset{(4)}{g}_{00} + \ldots, \tag{17.3}$$

$$g_{\alpha\beta} = \gamma_{\alpha\beta} + \overset{(2)}{g}_{\alpha\beta} + \overset{(4)}{g}_{\alpha\beta} + \ldots, \tag{17.4}$$

$$g_{0\alpha} = \overset{(3)}{g}_{0\alpha} + \overset{(5)}{g}_{0\alpha} + \ldots . \tag{17.5}$$

Here $\gamma_{\alpha\beta}$ is the spatial part of the Minkowski metric γ_{mn}, and the symbols $\overset{(k)}{g}_{mn}$ $(k = 2, 3, 4, \ldots)$ on the right-hand sides of (17.3)-(17.5) stand for terms of the order ε^k in the respective expansions of g_{mn}. Note that under time reversal, $t \to -t$, the sign of parameter ε must also be changed, whereby expansions (17.3) and (17.4) contain only even powers of ε and (17.5) only odd powers. The fact that $g_{0\alpha}$ does not contain $\overset{(1)}{g}_{0\alpha}$ seems natural since already the main (Newtonian!) approximation for $g_{0\alpha}$ must not be lower than the second order in ε.

Let us now find the expansions for $g = \det g_{mn}$ and g^{mn}. Using (17.3)-(17.5), we can show that

$$\begin{aligned} g = &-1 - \overset{(2)}{g}_{00} + \overset{(2)}{g}_{11} + \overset{(2)}{g}_{22} + \overset{(2)}{g}_{33} - \overset{(4)}{g}_{00} + \overset{(4)}{g}_{11} + \overset{(4)}{g}_{22} + \overset{(4)}{g}_{33} \\ &+ \overset{(2)}{g}_{00}(\overset{(2)}{g}_{11} + \overset{(2)}{g}_{22} + \overset{(2)}{g}_{33}) - \overset{(2)}{g}_{11}\overset{(2)}{g}_{22} - \overset{(2)}{g}_{11}\overset{(2)}{g}_{33} - \overset{(2)}{g}_{22}\overset{(2)}{g}_{33} \\ &+ \overset{(2)}{g}{}^2_{12} + \overset{(2)}{g}{}^2_{13} + \overset{(2)}{g}{}^2_{23} + \ldots, \end{aligned} \tag{17.6}$$

$$g^{00} = 1 + \overset{(2)}{g}{}^{00} + \overset{(4)}{g}{}^{00} + \ldots, \tag{17.7}$$

$$g^{\alpha\beta} = \gamma^{\alpha\beta} + \overset{(2)}{g}{}^{\alpha\beta} + \overset{(4)}{g}{}^{\alpha\beta} + \ldots, \tag{17.8}$$

$$g^{0\alpha} = \overset{(3)}{g}{}^{0\alpha} + \overset{(5)}{g}{}^{0\alpha} + \ldots, \tag{17.9}$$

where the $\overset{(k)}{g}{}^{mn}$ are expressed in terms of the $\overset{(k)}{g}_{mn}$ thus:

$$\overset{(2)}{g}{}^{00} = -\overset{(2)}{g}_{00},\quad \overset{(2)}{g}{}^{\alpha\beta} = -\overset{(2)}{g}_{\sigma\tau}\gamma^{\sigma\alpha}\gamma^{\tau\beta},\quad \overset{(3)}{g}{}^{0\alpha} = -\overset{(3)}{g}_{0\beta}\gamma^{\alpha\beta},$$
$$\overset{(4)}{g}{}^{00} = \overset{(2)}{g}{}^{2}_{00} - \overset{(4)}{g}_{00},\quad \overset{(4)}{g}{}^{\alpha\beta} = -\gamma^{\alpha\sigma}\gamma^{\beta\tau}\overset{(4)}{g}_{\sigma\tau} + \gamma^{\alpha\omega}\gamma^{\beta\sigma}\gamma^{\tau\lambda}\overset{(2)}{g}_{\omega\lambda}\overset{(2)}{g}_{\sigma\tau}, \tag{17.10}$$
$$\overset{(5)}{g}{}^{0\alpha} = -\gamma^{\alpha\beta}\overset{(5)}{g}_{0\beta} + \gamma^{\alpha\beta}\overset{(2)}{g}_{00}\overset{(3)}{g}_{0\beta} + \gamma^{\alpha\tau}\gamma^{\sigma\beta}\overset{(3)}{g}_{0\sigma}\overset{(2)}{g}_{\tau\beta}.$$

To write Eq. (8.37) for the terms in expansion (17.7)-(17.9) we must first find $\tilde{g}^{mn}$. These expansions yield

$$\tilde{g}^{00} = 1 + \overset{(2)}{\tilde{g}}{}^{00} + \overset{(4)}{\tilde{g}}{}^{00} + \ldots, \tag{17.11}$$

$$\tilde{g}^{0\alpha} = \overset{(3)}{\tilde{g}}{}^{0\alpha} + \overset{(5)}{\tilde{g}}{}^{0\alpha} + \ldots, \tag{17.12}$$

$$\tilde{g}^{\alpha\beta} = \tilde{\gamma}^{\alpha\beta} + \overset{(2)}{\tilde{g}}{}^{\alpha\beta} + \overset{(4)}{\tilde{g}}{}^{\alpha\beta} + \ldots, \tag{17.13}$$

with

$$\overset{(2)}{\tilde{g}}{}^{00} = \overset{(2)}{g}{}^{00} + \frac{1}{2}\overset{(2)}{A},\quad \overset{(4)}{\tilde{g}}{}^{00} = \overset{(4)}{g}{}^{00} + \frac{1}{2}\overset{(2)}{g}{}^{00}\overset{(2)}{A} + \frac{1}{2}\left(\overset{(4)}{A} - \frac{1}{4}\overset{(2)}{A}{}^{2}\right),$$
$$\overset{(3)}{\tilde{g}}{}^{0\alpha} = \overset{(3)}{g}{}^{0\alpha},\quad \overset{(5)}{\tilde{g}}{}^{0\alpha} = \overset{(5)}{g}{}^{0\alpha} + \frac{1}{2}\overset{(3)}{g}{}^{0\alpha}\overset{(2)}{A},\quad \overset{(2)}{\tilde{g}}{}^{\alpha\beta} = \overset{(2)}{g}{}^{\alpha\beta} + \frac{1}{2}\gamma^{\alpha\beta}\overset{(2)}{A}, \tag{17.14}$$
$$\overset{(4)}{\tilde{g}}{}^{\alpha\beta} = \overset{(4)}{g}{}^{\alpha\beta} + \frac{1}{2}\overset{(2)}{g}{}^{\alpha\beta}\overset{(2)}{A} + \frac{1}{2}\gamma^{\alpha\beta}\left(\overset{(4)}{A} - \frac{1}{4}\overset{(2)}{A}{}^{2}\right),$$

where we have introduced the following notation:

$$\overset{(2)}{A} = \overset{(2)}{g}_{00} - \overset{(2)}{g}_{11} - \overset{(2)}{g}_{22} - \overset{(2)}{g}_{33}, \tag{17.15}$$

$$\overset{(4)}{A} = \overset{(4)}{g}_{00} - \overset{(4)}{g}_{11} - \overset{(4)}{g}_{22} - \overset{(4)}{g}_{33} - \overset{(2)}{g}_{00}\left(\overset{(2)}{g}_{11} + \overset{(2)}{g}_{22} + \overset{(2)}{g}_{33}\right)$$
$$+ \overset{(2)}{g}_{11}\overset{(2)}{g}_{22} + \overset{(2)}{g}_{11}\overset{(2)}{g}_{33} + \overset{(2)}{g}_{22}\overset{(2)}{g}_{33} - \overset{(2)}{g}{}^{2}_{12} - \overset{(2)}{g}{}^{2}_{13} - \overset{(2)}{g}{}^{2}_{23}. \tag{17.16}$$

In a Galilean reference frame Eq. (8.37) yields

$$\frac{1}{2}\partial_0\overset{(2)}{g}_{00} - \frac{1}{2}\gamma^{\alpha\beta}\partial_0\overset{(2)}{g}_{\alpha\beta} = -\gamma^{\alpha\beta}\partial_\alpha\overset{(3)}{g}_{0\beta}, \tag{17.17}$$

$$\frac{1}{2}\partial_\alpha\overset{(2)}{g}_{00} + \frac{1}{2}\gamma^{\sigma\tau}\partial_\alpha\overset{(2)}{g}_{\sigma\tau} = \gamma^{\sigma\tau}\partial_\tau\overset{(2)}{g}_{\alpha\sigma}, \tag{17.18}$$

$$\partial_0\left[\overset{(2)}{g}{}^{2}_{00} - \overset{(4)}{g}_{00} - \frac{1}{2}\overset{(2)}{g}_{00}\overset{(2)}{A} + \frac{1}{2}\left(\overset{(4)}{A} - \frac{1}{4}\overset{(2)}{A}{}^{2}\right)\right] = -\partial_\alpha\left(\overset{(5)}{g}{}^{0\alpha} + \frac{1}{2}\overset{(3)}{g}{}^{0\alpha}\overset{(2)}{A}\right), \tag{17.19}$$

$$\partial_\beta\left[\overset{(4)}{g}{}^{\alpha\beta} + \frac{1}{2}\overset{(2)}{g}{}^{\alpha\beta}\overset{(2)}{A} + \frac{1}{2}\gamma^{\alpha\beta}\left(\overset{(4)}{A} - \frac{1}{4}\overset{(2)}{A}{}^{2}\right)\right] = \partial_0\overset{(3)}{g}_{0\beta}\gamma^{\beta\alpha}. \tag{17.20}$$

Now let us write the system of equations (8.36) for the $\overset{(k)}{g}_{mn}$. First we find the expansion of tensor G^{p}_{mn} in powers of ε. Since in a Galilean reference frame $G^{p}_{mn} = \Gamma^{p}_{mn}$, where

$$\Gamma^{p}_{mn} = \frac{1}{2}g^{pq}\left(\partial_m g_{qn} + \partial_n g_{qm} - \partial_q g_{mn}\right),$$

in view of (17.3)-(17.5) and (17.7)-(17.9) we find that

$$\Gamma^0_{00}=\frac{1}{2}\,\partial_0\overset{(2)}{g}_{00}+\frac{1}{2}\,(\overset{(2)}{g}{}^{00}\partial_0\overset{(2)}{g}_{00}-\overset{(3)}{g}{}^{0\alpha}\partial_\alpha\overset{(2)}{g}_{00})+\ldots,$$

$$\Gamma^0_{0\alpha}=\frac{1}{2}\,\partial_\alpha\overset{(2)}{g}_{00}+\frac{1}{2}\,(\partial_\alpha\overset{(4)}{g}_{00}+\overset{(2)}{g}{}^{00}\partial_\alpha\overset{(2)}{g}_{00})+\ldots,$$

$$\Gamma^0_{\alpha\beta}=\frac{1}{2}\,(\partial_\alpha\overset{(3)}{g}_{0\beta}+\partial_\beta\overset{(3)}{g}_{0\alpha}-\partial_0\overset{(2)}{g}_{\alpha\beta})+\ldots,\tag{17.21}$$

$$\Gamma^\alpha_{00}=-\frac{1}{2}\,\gamma^{\alpha\beta}\partial_\beta\overset{(2)}{g}_{00}-\frac{1}{2}\,\gamma^{\alpha\beta}\partial_\beta\overset{(4)}{g}_{00}+\gamma^{\alpha\beta}\partial_0\overset{(3)}{g}_{0\beta}-\frac{1}{2}\,\overset{(2)}{g}{}^{\alpha\beta}\partial_\beta\overset{(2)}{g}_{00}+\ldots.$$

$$\Gamma^\alpha_{0\beta}=\frac{1}{2}\,\gamma^{\alpha\sigma}\partial_\beta\overset{(3)}{g}_{\sigma 0}+\frac{1}{2}\,\gamma^{\alpha\sigma}\partial_0\overset{(2)}{g}_{\beta\sigma}-\frac{1}{2}\,\gamma^{\alpha\sigma}\partial_\sigma\overset{(3)}{g}_{0\beta}+\ldots,$$

$$\Gamma^\alpha_{\beta\omega}=\frac{1}{2}\,\gamma^{\alpha\sigma}(\partial_\beta\overset{(2)}{g}_{\sigma\omega}+\partial_\omega\overset{(2)}{g}_{\sigma\beta}-\partial_\sigma\overset{(2)}{g}_{\beta\omega})+\ldots.$$

On the basis of all this we can find the sought expansion for the second-rank curvature tensor R_{mn}. Since in Galilean coordinates $G^p_{mn}=\Gamma^p_{mn}$, after relatively simple calculations involving (8.31) we get

$$\begin{aligned}R_{00}=&-\frac{1}{2}\,\gamma^{\alpha\beta}\partial_\alpha\partial_\beta\overset{(2)}{g}_{00}-\frac{1}{2}\,\gamma^{\alpha\beta}\partial_\alpha\partial_\beta\overset{(4)}{g}_{00}+\gamma^{\alpha\beta}\partial_0\partial_\alpha\overset{(3)}{g}_{0\beta}\\&-\frac{1}{2}\,\partial_\alpha\,(\overset{(2)}{g}{}^{\alpha\beta}\partial_\beta\overset{(2)}{g}_{00})-\frac{1}{2}\,\gamma^{\alpha\beta}\partial_0\partial_0\overset{(2)}{g}_{\alpha\beta}\\&-\frac{1}{4}\,\gamma^{\alpha\beta}\partial_\beta\overset{(2)}{g}_{00}\gamma^{\sigma\tau}\partial_\alpha\overset{(2)}{g}_{\sigma\tau}+\frac{1}{4}\,\gamma^{\alpha\beta}\partial_\alpha\overset{(2)}{g}_{00}\partial_\beta\overset{(2)}{g}_{00}+\ldots,\end{aligned}\tag{17.22}$$

$$\begin{aligned}R_{0\alpha}=&\frac{1}{2}\,\gamma^{\beta\sigma}\partial_0\partial_\beta\overset{(2)}{g}_{\alpha\sigma}-\frac{1}{2}\,\gamma^{\beta\sigma}\partial_0\partial_\alpha\overset{(2)}{g}_{\beta\sigma}+\frac{1}{2}\,\gamma^{\beta\sigma}\partial_\alpha\partial_\beta\overset{(3)}{g}_{\sigma 0}\\&-\frac{1}{2}\,\gamma^{\beta\sigma}\partial_\beta\partial_\sigma\overset{(3)}{g}_{0\alpha}+\ldots,\end{aligned}\tag{17.23}$$

$$\begin{aligned}R_{\alpha\beta}=&-\frac{1}{2}\,\gamma^{\sigma\tau}\partial_\sigma\partial_\tau\overset{(2)}{g}_{\alpha\beta}+\frac{1}{2}\,\gamma^{\sigma\tau}\partial_\sigma\partial_\alpha\overset{(2)}{g}_{\tau\beta}+\frac{1}{2}\,\gamma^{\sigma\tau}\partial_\sigma\partial_\beta\overset{(2)}{g}_{\tau\alpha}\\&-\frac{1}{2}\,\partial_\alpha\partial_\beta\overset{(2)}{g}_{00}-\frac{1}{2}\,\gamma^{\sigma\tau}\partial_\alpha\partial_\beta\overset{(2)}{g}_{\tau\sigma}+\ldots.\end{aligned}\tag{17.24}$$

Allowing for (17.17) and (17.18), we finally obtain

$$\begin{aligned}R_{00}=&-\frac{1}{2}\,\gamma^{\alpha\beta}\partial_\alpha\partial_\beta\overset{(2)}{g}_{00}-\frac{1}{2}\,\gamma^{\alpha\beta}\partial_\alpha\partial_\beta\overset{(4)}{g}_{00}-\frac{1}{2}\,\partial_0\partial_0\overset{(2)}{g}_{00}\\&+\frac{1}{2}\,\gamma^{\sigma\alpha}\gamma^{\tau\beta}\overset{(2)}{g}_{\sigma\tau}\partial_\alpha\partial_\beta\overset{(2)}{g}_{00}+\frac{1}{2}\,\gamma^{\alpha\beta}\partial_\alpha\overset{(2)}{g}_{00}\partial_\beta\overset{(2)}{g}_{00}+\ldots,\end{aligned}\tag{17.25}$$

$$R_{0\alpha}=-\frac{1}{2}\,\gamma^{\beta\sigma}\partial_\beta\partial_\sigma\overset{(3)}{g}_{0\alpha}+\ldots,\tag{17.26}$$

$$R_{\alpha\beta}=-\frac{1}{2}\,\gamma^{\sigma\tau}\partial_\sigma\partial_\tau\overset{(2)}{g}_{\alpha\beta}+\ldots.\tag{17.27}$$

To complete the construction of approximate RTG equations we still need to expand the energy-momentum tensor of matter in a power series in ε. For what

follows it proves expedient to have the following expansions of T^{mn}:

$$T^{00}=\overset{(0)}{T}{}^{00}+\overset{(2)}{T}{}^{00}+\ldots,$$
$$T^{0\alpha}=\overset{(1)}{T}{}^{0\alpha}+\overset{(3)}{T}{}^{0\alpha}+\ldots, \tag{17.28}$$
$$T^{\alpha\beta}=\overset{(2)}{T}{}^{\alpha\beta}+\overset{(4)}{T}{}^{\alpha\beta}+\ldots.$$

Combining (17.3)-(17.5) with (17.28), we arrive at the following formulas for T_{mn}:

$$T_{00}=\overset{(0)}{T}_{00}+\overset{(2)}{T}_{00}+\ldots,$$
$$T_{0\alpha}=\overset{(1)}{T}_{0\alpha}+\overset{(3)}{T}_{0\alpha}+\ldots, \tag{17.29}$$
$$T_{\alpha\beta}=\overset{(2)}{T}_{\alpha\beta}+\overset{(4)}{T}_{\alpha\beta}+\ldots,$$

where

$$\overset{(0)}{T}_{00}=\overset{(0)}{T}{}^{00},\quad \overset{(2)}{T}_{00}=\overset{(2)}{T}{}^{00}+2\overset{(2)}{g}_{00}\overset{(0)}{T}{}^{00},$$
$$\overset{(1)}{T}_{0\alpha}=\gamma_{\alpha\beta}\overset{(1)}{T}{}^{0\beta},\quad \overset{(3)}{T}_{0\alpha}=\overset{(3)}{g}_{0\alpha}\overset{(0)}{T}{}^{00}+(\overset{(2)}{g}_{\alpha\beta}+\gamma_{\alpha\beta}\overset{(2)}{g}_{00})\overset{(1)}{T}{}^{0\beta}, \tag{17.30}$$
$$\overset{(2)}{T}_{\alpha\beta}=\gamma_{\alpha\sigma}\gamma_{\beta\tau}\overset{(2)}{T}{}^{\sigma\tau}.$$

Since on the right-hand side of (8.36) we have the combination

$$S_{mn}=T_{mn}-\frac{1}{2}g_{mn}T, \tag{17.31}$$

on the basis of (17.29) and (17.30) we can easily find the series expansion for the components of S_{mn} in powers of ε:

$$S_{00}=\frac{1}{2}\overset{(0)}{T}{}^{00}+\frac{1}{2}(\overset{(2)}{T}{}^{00}+2\overset{(2)}{g}_{00}\overset{(0)}{T}{}^{00}-\gamma_{\alpha\beta}\overset{(2)}{T}{}^{\alpha\beta})+\ldots, \tag{17.32}$$

$$S_{0\alpha}=\gamma_{\alpha\beta}\overset{(1)}{T}{}^{0\beta}+\ldots, \tag{17.33}$$

$$S_{\alpha\beta}=-\frac{1}{2}\gamma_{\alpha\beta}\overset{(0)}{T}{}^{00}+\left(\gamma_{\alpha\sigma}\gamma_{\beta\tau}-\frac{1}{2}\gamma_{\alpha\beta}\gamma_{\sigma\tau}\right)\overset{(2)}{T}{}^{\sigma\tau}$$
$$-\frac{1}{2}(\gamma_{\alpha\beta}\overset{(2)}{T}{}^{00}+\gamma_{\alpha\beta}\overset{(2)}{g}_{00}\overset{(0)}{T}{}^{00}+\overset{(2)}{g}_{\alpha\beta}\overset{(0)}{T}{}^{00})+\ldots. \tag{17.34}$$

Substituting into Eq. (8.36) the expansions (17.25)-(17.27) and (17.32)-(17.34) just established, we obtain

$$\gamma^{\alpha\beta}\partial_\alpha\partial_\beta\overset{(2)}{g}_{00}=-8\pi G\overset{(0)}{T}{}^{00}, \tag{17.35}$$

$$\gamma^{\alpha\beta}\partial_\alpha\partial_\beta\overset{(4)}{g}_{00}+\partial_0\partial_0\overset{(2)}{g}_{00}-\gamma^{\sigma\alpha}\gamma^{\tau\beta}\overset{(2)}{g}_{\sigma\tau}\partial_\alpha\partial_\beta\overset{(2)}{g}_{00}-\gamma^{\alpha\beta}\partial_\alpha\overset{(2)}{g}_{00}\partial_\beta\overset{(2)}{g}_{00}$$
$$=-8\pi G(\overset{(2)}{T}{}^{00}+2\overset{(2)}{g}_{00}\overset{(0)}{T}{}^{00}-\gamma_{\alpha\beta}\overset{(2)}{T}{}^{\alpha\beta}), \tag{17.36}$$

$$\gamma^{\beta\sigma}\partial_\beta\partial_\sigma\overset{(3)}{g}_{0\alpha}=-16\pi G\gamma_{\alpha\beta}\overset{(1)}{T}{}^{0\beta}, \tag{17.37}$$

$$\gamma^{\sigma\tau}\partial_\sigma\partial_\tau\overset{(2)}{g}_{\alpha\beta}=8\pi G\gamma_{\alpha\beta}\overset{(0)}{T}{}^{00}. \tag{17.38}$$

For given $\overset{(0)}{T}{}^{00}$, $\overset{(2)}{T}{}^{00}$, $\overset{(1)}{T}{}^{0\beta}$, and $\overset{(2)}{T}{}^{\alpha\beta}$, the system of equations (17.35)-(17.38) completely determines the effective Riemannian metric g^{mn} in the Newtonian and post-Newtonian approximations.

Assuming that

$$\overset{(2)}{g}_{00} = -2U, \tag{17.39}$$

where U is the Newtonian potential, we find that (17.35) yields

$$\nabla^2 U = -4\pi G \overset{(0)}{T}{}^{00}. \tag{17.40}$$

If we assume that U vanishes at infinity, the solution to this equation can be represented as follows:

$$U = G \int \frac{d^3x' \overset{(0)}{T}{}^{00}(\mathbf{x}', t)}{|\mathbf{x}-\mathbf{x}'|}. \tag{17.41}$$

Similarly, (17.37) and (17.38) yield

$$\overset{(3)}{g}_{0\alpha} = -4G\gamma_{\alpha\beta} \int \frac{d^3x' \overset{(1)}{T}{}^{0\beta}(\mathbf{x}', t)}{|\mathbf{x}-\mathbf{x}'|}, \tag{17.42}$$

$$\overset{(2)}{g}_{\alpha\beta} = 2G\gamma_{\alpha\beta} \int \frac{d^3x' \overset{(0)}{T}{}^{00}(\mathbf{x}', t)}{|\mathbf{x}-\mathbf{x}'|} = 2\gamma_{\alpha\beta} U. \tag{17.43}$$

If we allow for (17.39), (17.40), and (17.43), Eq. (17.36) can be written in the form

$$\nabla^2 (\overset{(4)}{g}_{00} - 2U^2) = -2\partial_0^2 U + 8\pi G (\overset{(2)}{T}{}^{00} - \gamma_{\alpha\beta} \overset{(2)}{T}{}^{\alpha\beta}). \tag{17.44}$$

Since $\overset{(4)}{g}_{00}$ must vanish at infinity, (17.44) implies that

$$\overset{(4)}{g}_{00} = 2U^2 + \frac{1}{2\pi}\partial_0^2 \int \frac{d^3x' U(\mathbf{x}', t)}{|\mathbf{x}-\mathbf{x}'|} - 2G \int \frac{d^3x' (\overset{(2)}{T}{}^{00} - \gamma_{\alpha\beta}\overset{(2)}{T}{}^{\alpha\beta})}{|\mathbf{x}-\mathbf{x}'|}. \tag{17.45}$$

Note that in view of (17.17) and (17.43), U and $\overset{(3)}{g}_{0\beta}$ are linked as follows:

$$\partial_0 U = \frac{1}{4}\gamma^{\alpha\beta}\partial_\alpha \overset{(3)}{g}_{0\beta}. \tag{17.46}$$

We have arrived at the solutions to the RTG equations for the components of the effective metric g_{mn} of the Riemann space-time in the following orders:

g_{00} to within terms of the order ε^4,

$g_{\alpha\beta}$ to within terms of the order ε^2,

$g_{0\alpha}$ to within terms of the order ε^3.

As we will see shortly, such accuracy in determining g_{mn} is practically sufficient for describing all the gravitational experiments conducted within the solar system. Hence, we need only Eqs. (17.17) and (17.18) from the system of equations (17.17)-(17.20). Note that in view of (17.39) and (17.43) Eq. (17.18) is satisfied automatically.

Before we begin studying gravitational effects in the post-Newtonian approximation, we must select a "model" for matter. Let us assume that we are dealing with a perfect fluid. Then for T^{mn} we can take the expression for the energy-momen-

tum tensor for a perfect fluid:

$$T^{mn} = [(p + \rho(1 + \Pi)) u^n u^m - p g^{mn}]. \tag{17.47}$$

Here, as usual, by p, ρ, Π we denote the isotropic pressure, the density of the perfect fluid, and the specific self-energy, respectively, and u^n is the 4-vector of velocity.

For the energy-momentum tensor T^{mn} and the invariant density ρ we have the following relationships: the covariant conservation law

$$\nabla_m T^{mn} = \partial_n T^{mn} + \Gamma^m_{nk} T^{kn} + \Gamma^n_{nk} T^{mk} = 0 \tag{17.48}$$

and the covariant continuity equation

$$\frac{1}{\sqrt{-g}} \partial_n (\sqrt{-g}\, \rho u^n) = 0. \tag{17.49}$$

In the Newtonian approximation, that is, when gravitation is ignored,

$$u^0 = 1 + O(\varepsilon^2), \quad u^\alpha = v^\alpha (1 + O(\varepsilon^2)), \tag{17.50}$$

and therefore (17.47) yields

$$\overset{(0)}{T}{}^{00} = \rho(1 + O(\varepsilon^2)), \tag{17.51}$$

$$\overset{(0)}{T}{}^{\alpha\beta} = O(\varepsilon^2), \tag{17.52}$$

$$\overset{(1)}{T}{}^{0\alpha} = \rho v^\alpha (1 + O(\varepsilon^2)). \tag{17.53}$$

In deriving (17.51)-(17.53) we allowed for the fact that the ratio p/ρ, or the specific isotropic pressure, is of the order of ε^2.

If in Eqs. (17.48) and (17.49) we discard terms of the order higher than ε, we arrive at the following equation for ρ:

$$\partial_0 \rho + \partial_\alpha (\rho v^\alpha) = 0. \tag{17.54}$$

This implies that in the Newtonian approximation the total mass of an object is

$$M = \int \rho \, d^3x$$

and is conserved.

Combining (17.51) and (17.53) with (17.41), (17.42), and (17.43), we get

$$\overset{(2)}{g}_{00} = -2U, \quad \overset{(3)}{g}_{0\alpha} = 4\gamma_{\alpha\beta} V^\beta, \quad \overset{(2)}{g}_{\alpha\beta} = 2\gamma_{\alpha\beta} U, \tag{17.55}$$

with

$$U = G \int \frac{\rho(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} d^3x', \tag{17.56}$$

$$V^\beta = -G \int \frac{\rho(\mathbf{x}', t)\, v^\beta}{|\mathbf{x} - \mathbf{x}'|} d^3x'. \tag{17.57}$$

Hence, in the lowest order the metric coefficients g_{mn} of the effective Riemann space-time can be represented as follows:

$$g_{00} = 1 - 2U, \quad g_{0\alpha} = 4\gamma_{\alpha\beta} V^\beta, \quad g_{\alpha\beta} = \gamma_{\alpha\beta}(1 + 2U). \tag{17.58}$$

If we employ the metric coefficients in this approximation in Eqs. (17.48) and (17.49), we can find the components of the energy-momentum tensor of matter in the next approximation. But for this we must first find in the Newtonian approxi-

mation $\sqrt{-g}$, u^0, and the connection coefficients Γ^p_{mn}. In view of (17.58) and (17.21) we have

$$\sqrt{-g} = 1 + 2U, \quad u^0 = 1 + U - \frac{1}{2} v_\alpha v^\alpha, \tag{17.59}$$

$$\begin{gathered}\Gamma^0_{00} = -\partial_0 U,\ \Gamma^0_{0\alpha} = -\partial_\alpha U,\ \Gamma^\alpha_{00} = \gamma^{\alpha\beta}\ \partial_\beta U,\\ \Gamma^0_{\alpha\beta} = -\gamma_{\alpha\beta}\partial_0 U + 2(\gamma_{\beta\sigma}\partial_\alpha + \gamma_{\alpha\sigma}\partial_\beta) V^\sigma,\\ \Gamma^\alpha_{0\beta} = 2\partial_\beta V^\alpha + \delta^\alpha_\beta \partial_0 U - 2\gamma^{\alpha\sigma}\gamma_{\beta\tau}\partial_\sigma V^\tau,\\ \Gamma^\alpha_{\beta\omega} = \delta^\alpha_\omega \partial_\beta U + \delta^\alpha_\beta \partial_\omega U - \gamma^{\alpha\sigma}\gamma_{\beta\omega}\partial_\sigma U.\end{gathered} \tag{17.60}$$

Then the covariant conservation law (17.48) will assume the form

$$\partial_0 \overset{(2)}{T}{}^{00} + \partial_\alpha \overset{(3)}{T}{}^{0\alpha} - \rho\partial_0 U - 2\rho v^\alpha \partial_\alpha U = O(\varepsilon^5), \tag{17.61}$$

$$\partial_\beta \overset{(2)}{T}{}^{\alpha\beta} + \partial_0(\rho v^\alpha) + \gamma^{\alpha\beta}\rho\partial_\beta U = O(\varepsilon^4) \tag{17.62}$$

and the continuity equation (17.49) the form

$$\frac{1}{\sqrt{-g}}\left[\partial_0\left(\rho + 3U\rho - \frac{1}{2}\rho v_\alpha v^\alpha\right) + \partial_\alpha\left(\rho v^\alpha + 3\rho v^\alpha U + \frac{1}{2}\rho v^\alpha v^2\right)\right] = O(\varepsilon^4). \tag{17.63}$$

To these equations we must add the equation of motion of a perfect fluid (Fock, 1939, 1959),

$$\hat{\rho}(\partial_0 v^\alpha + v^\beta \partial_\beta v^\alpha) = \gamma^{\alpha\beta}(-\hat{\rho}\partial_\beta U + \partial_\beta p) + \rho O(\varepsilon^4), \tag{17.64}$$

$$\hat{\rho}(\partial_0 \Pi + v^\beta \partial_\beta \Pi) = -p\partial_\alpha v^\alpha + \rho O(\varepsilon^5), \tag{17.65}$$

with

$$\hat{\rho} \equiv \sqrt{-g}\,\rho u^0. \tag{17.66}$$

According to (17.49), $\hat{\rho}$ is the mass density and is conserved.

In the required approximation we can write

$$\hat{\rho} = \rho\left(1 + 3U - \frac{1}{2} v_\alpha v^\alpha\right), \tag{17.67}$$

and therefore in (17.64) and (17.65) we can replace $\hat{\rho}$ with the invariant density ρ. From the system of equations (17.60)-(17.65) we can easily find the solutions for $\overset{(2)}{T}{}^{00}$, $\overset{(3)}{T}{}^{0\alpha}$, and $\overset{(2)}{T}{}^{\alpha\beta}$. These have the form

$$\begin{aligned}\overset{(2)}{T}{}^{00} &= \rho(2U + \Pi - v_\alpha v^\alpha),\\ \overset{(3)}{T}{}^{0\alpha} &= \rho v^\alpha(2U + \Pi - v_\beta v^\beta) + p v^\alpha,\\ \overset{(2)}{T}{}^{\alpha\beta} &= \rho v^\alpha v^\beta - \gamma^{\alpha\beta} p.\end{aligned} \tag{17.68}$$

Hence, (17.45) yields the following formula for $\overset{(4)}{g}_{00}$:

$$\overset{(4)}{g}_{00} = 2U^2 + \frac{1}{2\pi}\partial_0^2 \int \frac{d^3x' U(\mathbf{x}, t)}{|\mathbf{x} - \mathbf{x}'|} - 4\Phi_1 - 4\Phi_2 - 2\Phi_3 - 6\Phi_4, \tag{17.69}$$

where

$$\begin{gathered}\Phi_1 = -G\int \frac{\rho v_\alpha v^\alpha}{|\mathbf{x} - \mathbf{x}'|} d^3x', \quad \Phi_2 = G\int \frac{\rho U}{|\mathbf{x} - \mathbf{x}'|} d^3x',\\ \Phi_3 = G\int \frac{\rho\Pi}{|\mathbf{x} - \mathbf{x}'|} d^3x', \quad \Phi_4 = G\int \frac{p}{|\mathbf{x} - \mathbf{x}'|} d^3x'\end{gathered} \tag{17.70}$$

are known as the generalized gravitational potentials.
Since (see Vladimirov, 1984)

$$\frac{1}{2\pi}\int\frac{U\,d^3x'}{|\mathbf{x}-\mathbf{x}'|}=\frac{G}{2\pi}\int\rho(\mathbf{x}'',t)\,d^3x''\int\frac{d^3x'}{|\mathbf{x}-\mathbf{x}'|\,|\mathbf{x}'-\mathbf{x}''|}$$
$$=-G\int\rho(x'',t)\,|\mathbf{x}-\mathbf{x}''|\,d^3x'',$$

the final expression for $\overset{(4)}{g}_{00}$ proves to be

$$\overset{(4)}{g}_{00}=2U^2-G\partial_0^2\int\rho(\mathbf{x}',t)\,|\mathbf{x}-\mathbf{x}'|\,d^3x'-4\Phi_1-4\Phi_2-2\Phi_3-6\Phi_4. \quad (17.71)$$

Combining the formulas (17.58) and (17.71) for the metric coefficients g_{mn} of the tensor of the effective Riemann space-time, we obtain, to within the post-Newtonian approximation, the following:

$$g_{00}=1-2U+2U^2-G\partial_0^2\int\rho(\mathbf{x}',t)\,|\mathbf{x}-\mathbf{x}'|\,d^3x'$$
$$-4\Phi_1-4\Phi_2-2\Phi_3-6\Phi_4+O(\varepsilon^6), \quad (17.72)$$
$$g_{0\alpha}=4\gamma_{\alpha\beta}V^\beta+O(\varepsilon^5), \quad (17.73)$$
$$g_{\alpha\beta}=\gamma_{\alpha\beta}(1+2U)+O(\varepsilon^4). \quad (17.74)$$

Until recently the demands made on the various theories of gravitation were reduced to the necessity of obtaining Newton's law of universal gravitation in the weak-field limit and of describing three effects accessible to observation, namely, gravitational redshift in the Sun's field, the bending of a beam of light passing in the neighborhood of the Sun, and the Mercury perihelion shift. Insufficient accuracy of measurements in these experiments and the small body of experimental data explain why we now have a large number of theories of gravitation that provide a successful explanation for all of these effects.

To test these theories it is necessary, on the one hand, to increase the accuracy of measurements in the old experiments and suggest new experiments and, on the other, to develop appropriate theoretical tools, since the present requirements imposed on the theories of gravitation are clearly insufficient because of the large number of theories meeting these requirements.

Lately, with the development of relevant experimental techniques, especially astronautics, and the increase in accuracy of measurements, new possibilities have emerged for more precise measurement of the parameters of the orbits of various planets (primarily the Moon) and measurement of the time delay of radio signals in the Sun's gravitational field, and for new solar-system experiments. These experiments will enable narrowing still further the range of viable theories of gravitation. To facilitate comparison of the results of experiments conducted within the solar system with the predictions of the various theories of gravitation in which the Riemannian geometry for the motion of matter is the natural geometry, Nordtvedt and Will, 1972, and Will, 1978, 1981, developed what became known as the parametrized post-Newtonian (PPN) formalism (for the history of the PPN formalism see Misner, Thorne, and Wheeler, 1973 (p. 1049)).

In this formalism the Riemann-space-time metric generated by an object consisting of a perfect fluid is written as the sum of various generalized gravitational potentials with arbitrary coefficients known as post-Newtonian parameters. Using modified Nordtvedt-Will parameters, we can write the Riemann-space-time metric

as follows:

$$g_{00}=1-2U+2\beta U^2-(2\gamma+2+\alpha_3+\xi_1)\,\Phi_1+\xi_1 A$$
$$+2\xi_w\Phi_w-2\,[(3\gamma+1-2\beta+\xi_2)\,\Phi_2+(1+\xi_3)\,\Phi_3+3\,(\gamma+\xi_4)\,\Phi_4]$$
$$-(\alpha_1-\alpha_2-\alpha_3)\,w^\alpha w_\alpha U+\alpha_2 w^\alpha w^\beta U_{\alpha\beta}-(2\alpha_3-\alpha_1)\,w^\alpha V_\alpha, \tag{17.75}$$

$$g_{0\alpha}=\frac{1}{2}\,(4\gamma+3+\alpha_1-\alpha_2+\xi_1)\,\gamma_{\alpha\beta}V^\beta+\frac{1}{2}\,(1+\alpha_2-\xi_1)\,N_\alpha$$
$$+\frac{1}{2}\,(\alpha_1-2\alpha_2)\,w_\alpha\,U+\alpha_2 w^\beta U_{\alpha\beta}, \tag{17.76}$$

$$g_{\alpha\beta}=\gamma_{\alpha\beta}\,(1+2\gamma U), \tag{17.77}$$

where the w^α are the spatial components of the velocity of a reference frame with respect to a certain universal rest frame of reference (for some theories of gravitation this is the velocity of the center of mass of the solar system with respect to the reference frame in which the universe is at rest). In formulas (17.75)-(17.76), in addition to the generalized gravitational potentials introduced earlier in (17.56) (17.57), and (17.70), we have introduced the following potentials:

$$N_\alpha=\gamma_{\alpha\sigma}G\int\frac{\rho v_\beta\,(x^\beta-x'^\beta)\,(x^\sigma-x'^\sigma)}{|\,\mathbf{x}-\mathbf{x}'\,|^3}\,d^3x', \tag{17.78}$$

$$A=G\int\frac{\rho v_\alpha v_\beta\,(x^\alpha-x'^\alpha)\,(x^\beta-x'^\beta)}{|\,\mathbf{x}-\mathbf{x}'\,|^3}\,d^3x', \tag{17.79}$$

$$U_{\alpha\beta}=G\int\frac{\rho\,(x_\alpha-x'_\alpha)\,(x_\beta-x'_\beta)}{|\,\mathbf{x}-\mathbf{x}'\,|^3}\,d^3x', \tag{17.80}$$

$$\Phi_w=G\int\frac{\rho\,(\mathbf{x}',t)\,\rho\,(\mathbf{x}'',t)}{|\,\mathbf{x}-\mathbf{x}'\,|^3}\left\{\left[\frac{\mathbf{x}'-\mathbf{x}''}{|\,\mathbf{x}-\mathbf{x}''\,|}-\frac{\mathbf{x}-\mathbf{x}''}{|\,\mathbf{x}'-\mathbf{x}''\,|}\right]\cdot(\mathbf{x}-\mathbf{x}')\right\}d^3x'\,d^3x''. \tag{17.81}$$

To each theory of gravitation for which the Riemannian geometry is the natural geometry for describing the motion of matter there corresponds a set of ten values of the post-Newtonian parameters β, γ, α_1, α_2, α_3, ξ_1, ξ_2, ξ_3, ξ_4, and ξ_w. From the point of view of solar-system experiments, one theory of gravitation differs from another only by the values of these parameters. Note that in order to compare the various theories of gravitation the metric tensor g_{mn} of each theory must be written in terms of the coordinates used in writing the components (17.75)-(17.77), since otherwise comparison of the post-Newtonian parameters loses all meaning. Hence, after determining the metric g_{mn} generated by the gravitational field of the solar system we must change to a "canonical" coordinate system in which the metric tensor g_{mn} assumes the form (17.75)-(17.77).

A characteristic feature of the standard post-Newtonian expansion (17.75)-(17.77) is the fact that in "canonical" coordinates the off-diagonal components of the spatial part of the metric tensor g_{mn} vanish, while the nonzero components of g_{mn} do not contain terms of the type

$$\partial_0^2\int\rho\,|\,\mathbf{x}-\mathbf{x}'\,|\,d^3x'. \tag{17.82}$$

Our solutions (17.72)-(17.74) for g_{mn} contain, in contrast to (17.75)-(17.77), a term of type (17.82) in g_{00}. Hence, we must change to the coordinate system in which solutions (17.72)-(17.74) assume the form (17.75)-(17.77).

Performing the coordinate transformation

$$x'^0=x^0+\xi^0\,(x),\quad x'^\alpha=x^\alpha, \tag{17.83}$$

with $\xi^0(x) \simeq O(\varepsilon^3)$, we arrive at the following formulas for "transformed" metric coefficients:

$$g'_{00} = g_{00} + 2\partial_0\xi_0, \quad g'_{0\alpha} = g_{0\alpha} + \partial_\alpha\xi_0, \quad g'_{\alpha\beta} = g_{\alpha\beta}. \tag{17.84}$$

Selecting $\xi_0(x)$ in the form $\xi_0(x) = (G/2)\,\partial_0 \int \rho(\mathbf{x}', t)\,|\,\mathbf{x}-\mathbf{x}'\,|\,d^3x'$ and combining (17.84) with (17.72)-(17.74), we find the following expressions for the metric coefficients g'_{mn} in the canonical system of coordinates:

$$g'_{00} = 1 - 2U + 2U^2 - 4\Phi_1 - 4\Phi_2 - 2\Phi_3 - 6\Phi_4 + O(\varepsilon^6), \tag{17.85}$$

$$g'_{0\alpha} = \frac{7}{2}\gamma_{\alpha\beta}V^\beta - \frac{1}{2}N_\alpha + O(\varepsilon^5), \tag{17.86}$$

$$g'_{\alpha\beta} = \gamma_{\alpha\beta}(1+2U), \tag{17.87}$$

where in deriving (17.86) we employed the identity

$$\partial_\alpha\xi_0 \equiv -\frac{1}{2}(\gamma_{\alpha\beta}V^\beta - N_\alpha). \tag{17.88}$$

Comparing formulas (17.85)-(17.87) obtained in the post-Newtonian approximation for the metric coefficients g_{mn} in the RTG framework with formulas (17.75)-(17.77), we arrive at the following values for the post-Newtonian parameters:

$$\gamma = 1, \quad \beta = 1, \quad \alpha_1 = \alpha_2 = \alpha_3 = \xi_1 = \xi_2 = \xi_3 = \xi_4 = \xi_w = 0. \tag{17.89}$$

We note in passing that when the source of the gravitational field is a spherically symmetric object of radius r_0 the metric (17.85)-(17.87) assumes the form

$$g'_{00} = 1 - \frac{2MG}{r} + \frac{2M^2G^2}{r^2} + O\left(\frac{M^3G^3}{r^3}\right), \quad g'_{0\alpha} = 0, \tag{17.90}$$

$$g'_{\alpha\beta} = \gamma_{\alpha\beta}\left(1 + \frac{2MG}{r}\right) + O\left(\frac{M^2G^2}{r^2}\right),$$

where the total mass M of the source is defined thus:

$$M = 4\pi \int_0^{r_0} \rho\,[1 + \Pi + 3p/\rho + 2U]\,r^2\,dr.$$

To establish the theories of gravitation that in the post-Newtonian approximation enable us to describe all solar-system experiments it is sufficient to determine on the basis of all of these experiments the values of the ten post-Newtonian parameters and to select only those theories whose post-Newtonian approximation leads to values of parameters coinciding with those obtained in experiments. Then these theories of gravitation will be indistinguishable from the viewpoint of any experiments conducted with post-Newtonian accuracy. Further selection of theories of gravitation will be related either to increasing the accuracy still further or to seeking possible ways of studying gravitational waves and studying various phenomena in high gravitational fields.

As shown in Will, 1971b, the fact that the parameters α_1, α_2, and α_3 are equal to zero has a specific physical meaning: each theory of gravitation in which $\alpha_1 = \alpha_2 = \alpha_3 = 0$ does not contain a preferred universal rest frame of reference in the post-Newtonian limit. In this case, on passing from the universal rest frame of reference to a moving frame the metric of the effective Riemann space-time in the post-Newtonian limit is form-invariant, and the velocity w^α of the new system of coordinates with respect to the universal rest frame will not enter into the metric explicitly. Since (17.89) is valid, RTG belongs to such theories.

The linear dependence of parameters ξ and α also carries a certain physical meaning. As shown in Lee, Lightman, and Ni, 1974, if

$$\alpha_1 = 0, \quad \xi_3 = 0, \quad \alpha_2 - \xi_1 - 2\xi_w = 0, \quad \xi_2 = \xi_w, \qquad (17.91)$$
$$\alpha_3 + \xi_1 + 2\xi_w = 0, \quad 3\xi_4 + 2\xi_w = 0, \quad \xi_1 + 2\xi_w = 0,$$

the post-Newtonian equations of motion make it possible to determine quantities that are time independent in the post-Newtonian approximation. It has been demonstrated, however, (see Denisov, Logunov, Mestvirishvili, and Chugreev, 1985) that, generally speaking, it is possible to interpret these quantities as energy-momentum and angular momentum (i.e. as integrals of motion) only in theories of gravitation that contain the law of conservation of energy-momentum tensor of matter and gravitational field taken together. For example, in GR relationships (17.91) are valid but a detailed analysis shows that the quantities that are time independent in the post-Newtonian approximation are not the integrals of motion of the system consisting of matter and gravitational field.

Within the framework of RTG an isolated system is characterized in the pseudo-Euclidean space-time by all ten laws of conservation in the usual sense, laws that in the post-Newtonian approximation result in ten integrals of motion of the system. The fact that in RTG the relationships in (17.91) are valid corroborates this conclusion.

Chapter 18. RTG and Solar-System Gravitational Experiments. Ambiguities in the Predictions of GR

In this chapter we demonstrate that the predictions made in RTG concerning gravitational effects are unambiguous and agree with the known body of experimental data and that the respective predictions of GR are ambiguous.

The subject matter in this chapter is organized as follows. First we consider the standard effects: the equality of the inertial and gravitational masses, the deflection of light and radio signals by the gravitational field of the Sun, and the Mercury perihelion shift. Next we analyze such effects as the time delay of radio signals in the gravitational field of the Sun, the period of revolution of a test body in orbit, Shirokov's effect, and the precession of a gyroscope moving along a circular orbit in gravity. In some of these effects the ambiguity of the GR solutions manifests itself in the very first order in the constant of gravitational interaction, while in the other effects the ambiguity manifests itself in the second order.

18.1 Equality of the Inertial and Gravitational Masses in RTG

Gravitational experiments have established that the deviation from unity of the ratio of gravitational mass to inertial mass for objects of laboratory dimensions does not exceed one part in 10^{12} (see Braginsky and Panov, 1971). However, this does not mean that the gravitational and inertial masses of an object whose dimensions are great coincide with the same accuracy. For objects of laboratory dimensions the ratio of the gravitational self-energy to the total energy is no greater in order of magnitude than 10^{-25}. Hence, with an accuracy of one part in 10^{12} it is impossible to say how the gravitational self-energy is distributed between the inertial and gravitational masses in objects of laboratory dimensions.

The ratio of the inertial mass to the gravitational mass for extended objects for which the ratio of the gravitational self-energy to the total energy considera-

bly exceeds 10^{-25} will be discussed in Chapter 20. Here we consider only the general corollaries to which RTG and GR lead in connection with this problem.

In GR, as shown in Chapter 3, the value of the inertial mass depends on the choice of coordinate axes in three-dimensional space, which physically is meaningless. Therefore, it does not follow from GR that inertial mass is equal to the active gravitational mass. On the other hand, from RTG it follows that *the inertial mass and the active gravitational mass of an object coincide.*

Indeed, since the special principle of relativity forms the basis for RTG, the inertial mass of an island system is strictly defined and is given by the following formula:

$$m_1 = \int d^3x\,(t^{00}_{(g)} + t^{00}_{(M)}). \tag{18.1}$$

In view of conservation of the total energy-momentum tensor in the Minkowski space-time, or $\partial_p\,(t^{pn}_{(g)} + t^{pn}_{(M)}) = 0$, it is obvious that m_1 is time independent. Note also that (18.1) is a scalar with respect to the transformation of spatial coordinates.

Now let us write Eq. (8.28) for field $\widetilde{\Phi}^{00}$ in Cartesian coordinates. Since in this case $\widetilde{\Phi}^{00} = \Phi^{00}$, we have

$$\Box\Phi^{00} = 16\pi\,(t^{00}_{(g)} + t^{00}_{(M)}).$$

As is known, far from the source this equation may lead to a solution of the type r^{-1} const only if $t^{00}_{(g)} + t^{00}_{(M)}$ is time independent or very weakly dependent. In such conditions the equation for Φ^{00} assumes the form

$$\nabla^2\Phi^{00} = -16\pi\,(t^{00}_{(g)} + t^{00}_{(M)}),$$

whose solution will be represented as follows:

$$\Phi^{00} = 4\int \frac{d^3x'\,(t^{00}_{(g)} + t^{00}_{(M)})}{|\mathbf{x} - \mathbf{x}'|}.$$

From this we find that

$$\Phi^{00} \simeq \frac{4m_1}{r} \quad \text{as} \quad |\mathbf{x}| = r \to \infty. \tag{18.2}$$

In view of (8.1), for $\widetilde{g}^{00}$ we have the following formula:

$$\widetilde{g}^{00} \simeq 1 + \frac{4m_1}{r}.$$

On the other hand, if we determine $\widetilde{g}^{00}$ from (17.11), (17.14), and (17.55), we get

$$\widetilde{g}^{00} \simeq 1 + 4U,$$

with U the Newtonian potential. Since (in the system of units where $c = G = 1$) we have $U = M/r$ far from the source, with M, by definition, the gravitational mass of the object, we arrive at the following identity:

$$m_1 = M, \tag{18.3}$$

which is what we set out to prove.

We note in passing that within the RTG framework the quantity

$$P^n = \int (t^{0n}_{(g)} + t^{0n}_{(M)})\,d^3x \tag{18.4}$$

is well-defined and constitutes the 4-vector of energy-momentum of the system. Similarly, the angular momentum of the system is also well-defined in RTG and is a tensor with respect to any coordinate transformations in the four-dimensional Minkowski space-time.

In GR, as noted in Chapter 3, the value of the inertial mass depends on the choice of coordinates in the three-dimensional space, a result meaningless from the standpoint of physics.

Now let us consider this problem in the light of the results obtained in Chapter 12. We will show that in view of the arbitrariness in the choice of solutions to the Hilbert-Einstein equations, the inertial mass (as defined in GR) may, generally speaking, assume any value and, therefore, there can be no equality between the inertial and gravitational masses in GR. As in Chapter 3, we base our calculations of the inertial mass (in the system of units in which $c = G = 1$) on formula (2.11) written in terms of Cartesian coordinates:

$$m_1 = \oint h^{00\alpha}(x)\, dS_\alpha, \tag{18.5}$$

where

$$h^{00\alpha}(x) = \frac{1}{16\pi} \frac{\partial}{\partial x^\beta} m^{\alpha\beta}(x). \tag{18.6}$$

In the last expression we have introduced the notation

$$m^{\alpha\beta}(x) = -gg^{00}\left(g^{\alpha\beta} - \frac{g^{0\alpha}g^{0\beta}}{g^{00}}\right). \tag{18.7}$$

Since with respect to three-dimensional (spatial) transformations g^{00} is a scalar,

$$g(x) = D^2 g(x'), \quad \text{where } D = \det\left(\frac{\partial x'^\nu}{\partial x^\mu}\right),$$

and $g^{\alpha\beta} - g^{0\alpha}g^{0\beta}/g^{00}$ is a second-rank tensor, the transformation law for $m^{\alpha\beta}(x)$ assumes the form

$$m^{\alpha\beta}(x) = D^2 \frac{\partial x^\alpha}{\partial x'^\nu} \frac{\partial x^\beta}{\partial x'^\mu} m^{\mu\nu}(x'), \tag{18.8}$$

where

$$m^{\mu\nu}(x') = -g(x')\, g^{00}(x')\left(g^{\mu\nu}(x') - \frac{g^{0\mu}(x')\, g^{0\nu}(x')}{g^{00}(x')}\right). \tag{18.9}$$

If for the x'^ν we take the spherical coordinates $x'^1 = r$, $x'^2 = \theta$, and $x'^3 = \varphi$, then on the basis of (12.41) with $A(r) = 0$ we arrive at the following formulas for the $m^{\mu\nu}(x')$:

$$\begin{gathered} m^{11}(x') = -W^2 \sin^2\theta, \\ m^{22}(x') = -\frac{W\sqrt{W}}{\sqrt{W} - 2M}\left(\frac{d\sqrt{W}}{dr}\right)^2 \sin^2\theta, \\ m^{33}(x') = -\frac{W\sqrt{W}}{\sqrt{W} - 2M}\left(\frac{d\sqrt{W}}{dr}\right)^2, \\ m^{\mu\nu}(x') = 0 \quad \text{if} \quad \mu \neq \nu. \end{gathered} \tag{18.10}$$

It can be shown that

$$D = \frac{1}{r^2 \sin\theta},$$

and, hence, combining (18.8) with (18.10), we get

$$m^{\alpha\beta}(x) = \frac{1}{r^4 \sin^2\theta}\left[\frac{\partial x^\alpha}{\partial x'^1}\frac{\partial x^\beta}{\partial x'^1} m^{11}(x') + \frac{\partial x^\alpha}{\partial x'^2}\frac{\partial x^\beta}{\partial x'^2} m^{22}(x') + \frac{\partial x^\alpha}{\partial x'^3}\frac{\partial x^\beta}{\partial x'^3} m^{33}(x')\right]. \tag{18.11}$$

Hence, taking into account the values of the elements of the transformation matrix

$$\frac{\partial x^\alpha}{\partial x'^\nu} = \begin{pmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ r\cos\theta\cos\varphi & r\cos\theta\sin\varphi & -r\sin\theta \\ -r\sin\theta\sin\varphi & r\sin\theta\cos\varphi & 0 \end{pmatrix}, \tag{18.12}$$

we arrive at

$$\begin{aligned} m^{11}(x) &= -B(r)\sin^2\theta\cos^2\varphi - C(r), \\ m^{22}(x) &= -B(r)\sin^2\theta\sin^2\varphi - C(r), \\ m^{33}(x) &= -B(r)\cos^2\theta - C(r), \\ m^{12}(x) &= -B(r)\sin^2\theta\cos\varphi\sin\varphi, \\ m^{13}(x) &= -B(r)\sin\theta\cos\theta\cos\varphi, \\ m^{23}(x) &= -B(r)\sin\theta\cos\theta\sin\varphi, \end{aligned} \tag{18.13}$$

where

$$B(r) = \frac{W^2}{r^4} - \frac{W}{r^2}\frac{\sqrt{W}}{\sqrt{W}-2M}\left(\frac{d\sqrt{W}}{dr}\right)^2, \tag{18.14}$$

$$C(r) = \frac{W}{r^2}\frac{\sqrt{W}}{\sqrt{W}-2M}\left(\frac{d\sqrt{W}}{dr}\right)^2. \tag{18.15}$$

To calculate the inertial mass m_1 of a static spherically symmetric object, we can use the formula

$$m_1 = -\frac{1}{16\pi}\lim_{r\to\infty} r\int x_\alpha \frac{\partial x'^\nu}{\partial x^\beta}\frac{\partial m^{\alpha\beta}(x)}{\partial x'^\nu}\sin\theta\, d\theta\, d\varphi, \tag{18.16}$$

which follows from (18.5) and (18.6). In (18.16) we have allowed for the fact that

$$dS_\alpha = -r x_\alpha \sin\theta\, d\theta\, d\varphi.$$

Since the transformation matrix $\partial x'^\nu/\partial x^\beta$ has the form

$$\frac{\partial x'^\nu}{\partial x^\beta} = \begin{pmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ r^{-1}\cos\theta\cos\varphi & r^{-1}\cos\theta\sin\varphi & -r^{-1}\sin\theta \\ -r^{-1}(\sin\varphi)/\sin\theta & r^{-1}(\cos\varphi)/\sin\theta & 0 \end{pmatrix}^{\mathrm{T}}, \tag{18.17}$$

on the basis of (18.13) we can easily calculate the integrand in (18.16) in spherical coordinates:

$$x_\alpha \frac{\partial x'^\nu}{\partial x^\beta}\frac{\partial m^{\alpha\beta}(x)}{\partial x'^\nu}\sin\theta = \left[r\frac{d}{dr}(B+C) + 2B\right]\sin\theta. \tag{18.18}$$

Integration in (18.18) with respect to angular variables yields the final formula for m_1:

$$m_1 = -\frac{1}{4} \lim_{r\to\infty} \left[r^2 \frac{d}{dr}\left(\frac{W^2}{r^4}\right) + 2rB\right]. \tag{18.19}$$

If for $W(r)$ we take the one-parameter family of solutions to the Hilbert-Einstein equations, or

$$W(r) = [r + (1+\lambda) M]^2, \tag{18.20}$$

where λ is any finite number, then from (18.19) it follows that

$$m_1 \equiv M \tag{18.21}$$

in the class of solutions (18.20).

In Chapter 12 it was demonstrated that the system of RTG equations for $W(r)$ admits of only one solution that coincides with (18.20) at $\lambda = 0$, with the result that in this system of equations, in view of (18.21), the inertial mass m_1 is always equal to the gravitational mass M. Note that for Schwarzschild's solution, which corresponds to $\lambda = -1$ in (18.20), the inertial mass m_1 also coincides with the gravitational mass M. On the other hand, in GR, $W(r)$ is an arbitrary function of variable r, whereby solutions (18.20) have no special meaning for GR. If one selects $W(r)$ such that for $r\to\infty$ the asymptotic behavior of $W(r)$ is

$$W(r) = r^2\left[1+\alpha^2\left(\frac{8M}{r}\right)^{1/2}\right]^2, \tag{18.22}$$

where α is any finite number, then on the basis of (18.14) and (18.19) we arrive at the following expression for the inertial mass m_1:

$$m_1 = M(1+\alpha^4). \tag{18.23}$$

We see that, in view of the arbitrariness in selecting α, the inertial mass m_1 may assume in GR any fixed value $m_1 \geqslant M$, which immediately implies that the energy of a system may also assume any value; hence, the results obtained by Ponomarev, 1985, are erroneous.

In conclusion we give the asymptotic expressions for the metric coefficients (12.41) as $r\to\infty$, when $A(r) = 0$ and $W(r)$ is given by (18.22):

$$\begin{aligned}
g_{00}(x') &\simeq 1 - \frac{2M}{r} + O\left(\frac{1}{r^{3/2}}\right),\\
g_{11}(x') &\simeq -\left[1+\alpha^2\left(\frac{8M}{r}\right)^{1/2}\right] + O\left(\frac{1}{r}\right),\\
g_{22}(x') &\simeq -r^2\left[1+2\alpha^2\left(\frac{8M}{r}\right)^{1/2}\right],\\
g_{33}(x') &\simeq -r^2\left[1+2\alpha^2\left(\frac{8M}{r}\right)^{1/2}\right]\sin^2\theta.
\end{aligned} \tag{18.24}$$

The reader can now easily see that

$$\lim_{r\to\infty} g_{mn}(x') = \gamma_{mn}(x'),$$

which implies that for the Riemann space-time with metric (18.24) there exists an asymptotically flat Minkowski space-time.

18.2 The Equations of Motion of a Test Body Along a Geodesic in the Sun's Gravitational Field

In calculating the standard effects occurring in the Sun's gravitational field the common approach is to take for an idealized model of the Sun a static spherically symmetric ball of radius R_0. The metric coefficients for objects possessing these properties were determined in Chapter 12. There it was also noted that the solutions to none but the Hilbert-Einstein equations in the exterior of a static spherically symmetric source contain two arbitrary functions, $A(r)$ and $W(r)$, and therefore GR cannot in principle give well-defined predictions concerning gravitational effects. On the other hand, the solutions to the system of RTG equations for the metric coefficients of the effective Riemann space-time are well-defined and unambiguous. Hence, in Chapter 12 we arrived at an important conclusion, namely, that there is no arbitrariness in RTG, and, therefore, that for the gravitational experiments considered below the predictions within the RTG framework are fully unambiguous.

Let us illustrate this with examples. To this end to describe gravitational effects in the solar system we take the metric coefficients to be

$$\begin{gathered} g_{00}(r) = U(r) = \frac{\sqrt{W(r)} - 2GM_\odot}{\sqrt{W(r)}}, \\ g_{11}(r) = -V(r) = -\frac{\sqrt{W(r)}}{\sqrt{W(r)} - 2GM_\odot}\left(\frac{d\sqrt{W(r)}}{dr}\right)^2, \\ g_{22}(r) = -W(r), \quad g_{33}(r, \theta) = -W(r)\sin^2\theta. \end{gathered} \tag{18.25}$$

These formulas transform from (12.41) if we set

$$A(r) = 0, \quad m = GM_\odot, \tag{18.26}$$

where $M_\odot$ is the Sun's active gravitational mass and G the gravitational constant. The condition $A(r) = 0$ does not change the essence of the problem and is chosen only to simplify matters.

Within the GR framework, $\sqrt{W(r)}$ is an arbitrary function of r, while in RTG it has the form $\sqrt{W(r)} = r + GM_\odot$. Note that in the exterior of the source

$$\sqrt{W(r)} > 2GM_\odot. \tag{18.27}$$

The equations of motion of a material particle or a photon are written in the form of geodesic equations in the effective Riemann space-time with metric (18.25):

$$\frac{d^2x^m}{d\sigma^2} + \Gamma^m_{pq}\frac{dx^p}{d\sigma}\frac{dx^q}{d\sigma} = 0. \tag{18.28}$$

In metric (18.25), the nonzero connection coefficients Γ^m_{pq} are

$$\begin{gathered} \Gamma^0_{01} = \frac{1}{2U}\frac{\partial U}{\partial r}, \quad \Gamma^1_{00} = \frac{1}{2V}\frac{\partial U}{\partial r}, \quad \Gamma^1_{11} = \frac{1}{2V}\frac{\partial V}{\partial r}, \\ \Gamma^1_{22} = -\frac{1}{2V}\frac{\partial W}{\partial r}, \quad \Gamma^1_{33} = \Gamma^1_{22}\sin^2\theta, \\ \Gamma^2_{12} = \Gamma^3_{13} = \frac{1}{2W}\frac{\partial W}{\partial r}, \quad \Gamma^2_{33} = -\sin\theta\cos\theta, \quad \Gamma^3_{23} = \cot\theta. \end{gathered} \tag{18.29}$$

For brevity of notation in (18.29) we have left the function $U(r)$ and $V(r)$, not replacing them with (12.40). In the final analysis, however, the connection coefficients (18.29) are expressed only in terms of $W(r)$ and θ.

Now let us write Eq. (18.28) explicitly. Combining (18.29) with (18.28), we get

$$\frac{d^2t}{d\sigma^2}+\frac{1}{U}\frac{dU}{dr}\frac{dt}{d\sigma}\frac{dr}{d\sigma}=0, \tag{18.30}$$

$$\frac{d^2r}{d\sigma^2}+\frac{1}{2V}\frac{dU}{dr}\left(\frac{dt}{d\sigma}\right)^2+\frac{1}{2V}\frac{dV}{dr}\left(\frac{dr}{d\sigma}\right)^2-\frac{1}{2V}\frac{dW}{dr}\left(\frac{d\theta}{d\sigma}\right)^2$$
$$-\frac{1}{2V}\frac{dW}{dr}\left(\frac{d\varphi}{d\sigma}\right)^2\sin^2\theta=0, \tag{18.31}$$

$$\frac{d^2\theta}{d\sigma^2}-\left(\frac{d\varphi}{d\sigma}\right)^2\sin\theta\cos\theta+\frac{1}{W}\frac{dW}{dr}\frac{dr}{d\sigma}\frac{d\theta}{d\sigma}=0, \tag{18.32}$$

$$\frac{d^2\varphi}{d\sigma^2}+\frac{1}{W}\frac{dW}{dr}\frac{dr}{d\sigma}\frac{d\varphi}{d\sigma}+2\frac{d\theta}{d\sigma}\frac{d\varphi}{d\sigma}\cot\theta=0. \tag{18.33}$$

Since the field is isotropic, without loss of generality we can consider only trajectories that lie in the equatorial plane. Hence, in Eqs. (18.30)-(18.33) we can set $\theta=\pi/2$. In this case Eq. (18.32) becomes an identity, while Eqs. (18.31) and (18.33) assume the form

$$\frac{d^2r}{d\sigma^2}+\frac{1}{2V}\frac{dU}{dr}\left(\frac{dt}{d\sigma}\right)^2+\frac{1}{2V}\frac{dV}{dr}\left(\frac{dr}{d\sigma}\right)^2-\frac{1}{2V}\frac{dW}{dr}\left(\frac{d\varphi}{d\sigma}\right)^2=0, \tag{18.34}$$

$$\frac{d^2\varphi}{d\sigma^2}+\frac{1}{W}\frac{dW}{dr}\frac{dr}{d\sigma}\frac{d\varphi}{d\sigma}=0. \tag{18.35}$$

Combining (18.30) with (18.35), we easily find that

$$\frac{dt}{d\sigma}=\frac{c}{U(r)}, \tag{18.36}$$

$$\frac{d\varphi}{d\sigma}=\frac{J}{W(r)}, \tag{18.37}$$

where c and J are constants of integration and also integrals of motion in the problem considered here. By redefining parameter σ we can always ensure that c in (18.36) is equal to unity. Hence, in what follows we use, without loss of generality, the following formula:

$$\frac{dt}{d\sigma}=\frac{1}{U(r)}. \tag{18.38}$$

Substituting (18.37) and (18.38) into Eqs. (18.34), we can write this equation as follows:

$$\frac{d^2r}{d\sigma^2}-\frac{1}{2V}\frac{d}{dr}\left(\frac{1}{U}\right)+\frac{1}{2V}\frac{dV}{dr}\left(\frac{dr}{d\sigma}\right)^2+\frac{J^2}{2V}\frac{d}{dr}\left(\frac{1}{W}\right)=0.$$

Multiplying this into $2V(dr/d\sigma)$ yields

$$\frac{d}{d\sigma}\left[V\left(\frac{dr}{d\sigma}\right)^2-\frac{1}{U(r)}+\frac{J^2}{W(r)}\right]=0,$$

which implies that

$$V\left(\frac{dr}{d\sigma}\right)^2-\frac{1}{U(r)}+\frac{J^2}{W(r)}=-E \tag{18.39}$$

is also an integral of motion in the given problem.

Let us establish how the proper time τ is linked to the parameter σ of the trajectory. We determine the proper time from the expression for the line element

$$d\tau^2=g_{00}\,dt^2+g_{11}\,dr^2+g_{22}\,d\theta^2+g_{33}\,d\varphi^2, \tag{18.40}$$

with the metric coefficients specified in (18.25). Since $\theta = \pi/2$, we can write (18.40) in the form

$$d\tau^2 = \left[U \left(\frac{dt}{d\sigma}\right)^2 - V \left(\frac{dr}{d\sigma}\right)^2 - W \left(\frac{d\varphi}{d\sigma}\right)^2 \right] d\sigma^2.$$

Allowing here for (18.37)-(18.39), we get

$$d\tau^2 = E\, d\sigma^2. \tag{18.41}$$

Since for massless particles $d\tau^2 = 0$, we conclude that for a photon

$$E = 0, \tag{18.42}$$

while if the rest mass of a particle is nonzero, (18.41) implies

$$E > 0. \tag{18.43}$$

Using (18.38) and (18.41), we can link the proper time τ and the temporal coordinate t in the Minkowski space-time:

$$d\tau^2 = EU^2\, dt^2. \tag{18.44}$$

Similarly, from (18.37) and (18.39), excluding parameter σ and using (18.38), we find that

$$\frac{d\varphi}{dt} = \frac{JU}{W}, \tag{18.45}$$

$$\frac{V}{U^2}\left(\frac{dr}{dt}\right)^2 - \frac{1}{U} + \frac{J^2}{W} = -E. \tag{18.46}$$

The solvability of Eq. (18.46) requires that

$$\frac{J^2}{W(r)} + E \leqslant \frac{1}{U(r)}. \tag{18.47}$$

Finding dt from (18.45) and substituting it into (18.46), we get

$$\frac{J^2V}{W^2}\left(\frac{dr}{d\varphi}\right)^2 - \frac{1}{U} + \frac{J^2}{W} + E = 0. \tag{18.48}$$

Combining this with (18.27) and (18.47), we finally get

$$\frac{d\varphi}{dr} = \pm \frac{V^{1/2}}{W}\left(\frac{1}{J^2U} - \frac{1}{W} - \frac{E}{J^2}\right)^{-1/2}. \tag{18.49}$$

18.3 Deflection of Light and Radio Waves in the Sun's Gravitational Field*

Suppose that we wish to describe the trajectories of particles coming in from outer regions of space and passing in the vicinity of the Sun. Let us place the origin of the coordinate system at the center of the Sun and assume that the particle in question moves in the equatorial plane xy parallel to the x axis from $+\infty$ to $-\infty$. It is convenient to describe the trajectory in terms of φ as a function of r. Since angle $\varphi(r)$ is reckoned from the positive direction of the x axis, it is obvious that $\varphi(+\infty) = 0$.

If the particle were not deflected by the Sun's gravitational field, the variation of angle φ over the entire motion would be π, since $\varphi(-\infty) - \varphi(+\infty) = \pi$.

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* See Logunov and Loskutov, 1986c, Logunov, Loskutov, and Chugreev, 1986, and Weinberg, 1972.

However, as a result of the action of the gravitational field the particle trajectory deviates from a straight line, and the measure of this deviation can be characterized by the quantity

$$\delta\varphi = \Delta\varphi - \pi. \tag{18.50}$$

To calculate $\Delta\varphi$ we write the solution to Eq. (18.49), which expresses φ in terms of r, in the form

$$\varphi = \pm \int dr \frac{V^{1/2}}{W} \left(\frac{1}{J^2 U} - \frac{1}{W} - \frac{E}{J^2} \right)^{-1/2}. \tag{18.51}$$

Combining this with formulas (18.25) for $U(r)$ and $V(r)$, we obtain

$$\varphi = \pm J \int \frac{d\sqrt{W}}{\{\sqrt{W}\,[W^{3/2}(1-E) - J^2(\sqrt{W} - 2GM_\odot) + 2EWGM_\odot]\}^{1/2}}. \tag{18.52}$$

This formula gives the shape of the particle trajectory in the Sun's gravitational field. Far from the Sun, that is, as $r \to \infty$, we have $W(r) \simeq r^2$, in view of (12.36). Hence, on the basis of (18.25) and (18.46) we have (as $r \to \infty$):

$$U(r) \to 1, \; V(r) \to 1, \; \left(\frac{dr}{dt}\right)^2 = 1 - E. \tag{18.53}$$

Since in this region of space the particle is in free motion, the particle's velocity $dr/dt = v$ is constant and, hence $E = \text{const}$, which completely agrees with the meaning of E as an integral of motion. Since for a photon (in the system of units where $c = 1$) $v = 1$, we have $E = 0$. For a material particle $v < 1$, and, therefore, $E < 1$.

Suppose that r_0 is the closest distance from the particle's trajectory. Then, because $(dr/d\varphi)_{r=r_0} = 0$, we have the following formula for the square of the integral of motion J:

$$J^2 = \frac{W_0^{3/2}}{\sqrt{W_0} - 2GM_\odot} - EW_0, \tag{18.54}$$

where we have used (18.48) and the formula that links U with W (see (18.25)) and have introduced the notation $W_0 = W(r_0)$. Substituting (18.54) into (18.52), we get

$$\varphi(r) = [W_0^{3/2} - EW_0(\sqrt{W_0} - 2GM_\odot)]^{1/2} \int\limits_{\sqrt{W(r)}}^{\infty} d\sqrt{W}\,\{\sqrt{W}\,[W^{3/2}(1-E) + 2EWGM_\odot]$$
$$\times (\sqrt{W_0} - 2GM_\odot) - [W_0^{3/2} - EW_0(\sqrt{W_0} - 2GM_\odot)](\sqrt{W} - 2GM_\odot)\}^{-1/2}. \tag{18.55}$$

Since we will be interested only in the deviation of light and radio waves in the Sun's gravitational field, in (18.55) we must set $E = 0$. In this case we have

$$\varphi(r) = W_0^{3/2} \int\limits_{\sqrt{W(r)}}^{\infty} d\sqrt{W}\,\{[W^{3/2}(\sqrt{W_0} - 2GM_\odot) - W_0^{3/2}(\sqrt{W} - 2GM_\odot)]\}^{-1/2}. \tag{18.56}$$

If we introduce the notation

$$W_1 = -\frac{1}{2}\sqrt{W_0}\left[1 + \sqrt{\frac{\sqrt{W_0} + 6GM_\odot}{\sqrt{W_0} - 2GM_\odot}}\right], \tag{18.57}$$

$$W_2 = -\frac{1}{2}\sqrt{W_0}\left[1 - \sqrt{\frac{\sqrt{W_0} + 6GM_\odot}{\sqrt{W_0} - 2GM_\odot}}\right], \tag{18.58}$$

we can write (18.56) in the form

$$\varphi(r)=J_0\int\limits_{\sqrt{W(r)}}^{\infty}\frac{d\sqrt{W}}{[\sqrt{W}(\sqrt{W}-\sqrt{W_0})(\sqrt{W}-W_1)(\sqrt{W}-W_2)]^{1/2}}. \quad (18.59)$$

The value of J_0 can be obtained from (18.54) if we put $E = 0$. Setting $r = r_0$ in (18.59) and integrating, we obtain

$$\varphi(r_0)=\frac{2J_0}{\sqrt{W_0}}\left[\frac{\sqrt{W_0}-2GM_\odot}{\sqrt{W_0}+6GM_\odot}\right]^{1/4}F(\nu,\ q), \quad (18.60)$$

where

$$\nu=\sin^{-1}\left(\frac{W_2-W_1}{\sqrt{W_0}-W_1}\right)^{1/2}, \quad (18.61)$$

$$q=\left(\frac{W_2}{\sqrt{W_0}}\ \frac{\sqrt{W_0}-W_1}{W_2-W_1}\right)^{1/2}, \quad (18.62)$$

and

$$F(\nu,\ q)=\int\limits_0^{\nu}(1-q^2\sin^2 t)^{-1/2}\,dt$$

is an elliptic integral of the first kind.

Since the total variation of angle $\varphi(r)$ when a photon travels from infinity to point $r = r_0$ and then goes on to infinity is $2\varphi(r_0)$, for $\Delta\varphi$ we have

$$\Delta\varphi=\frac{4J_0}{\sqrt{W_0}}\left(\frac{\sqrt{W_0}-2GM_\odot}{\sqrt{W_0}+6GM_\odot}\right)^{1/4}F(\nu,\ q). \quad (18.63)$$

In Appendix 4 we show that when $\sqrt{W_0}\gg 2GM_\odot$, from (18.63), in the second order in $2GM_\odot/\sqrt{W_0}$, there follows the expansion

$$\Delta\varphi\simeq\pi+\frac{4GM_\odot}{\sqrt{W_0}}+\frac{4(GM_\odot)^2}{W_0}\left(\frac{15}{16}\pi-1\right). \quad (18.64)$$

Substituting this into (18.50), we find that in the second order in $2GM_\odot/\sqrt{W_0}$

$$\delta\varphi\simeq\frac{4GM_\odot}{\sqrt{W_0}}+\frac{4(GM_\odot)^2}{W_0}\left(\frac{15}{16}\pi-1\right). \quad (18.65)$$

For r_0 we take the radius of the Sun, and the relationship between W_0 and r_0 will be assumed to be

$$W_0=[r_0+GM_\odot(1+\lambda)]^2, \quad (18.66)$$

where λ is an adjustable parameter. For all values of λ satisfying the condition

$$r_0\gg GM_\odot\,|\,1+\lambda\,| \quad (18.67)$$

to within the second order in $GM_\odot/r_0$ we have

$$\delta\varphi\simeq\frac{4GM_\odot}{r_0}-\frac{4(GM_\odot)^2}{r_0^2}\left[(2+\lambda)-\frac{15}{16}\pi\right]. \quad (18.68)$$

In GR the range of admissible values of parameter λ in (18.68) is limited only by condition (18.67), with the result that the prediction of GR concerning the deflection of light in the Sun's gravitational field contains, in the second order

in $GM_\odot/r_0$, an ambiguity. In RTG, on the other hand, parameter λ may assume only one value, $\lambda \equiv 0$, and therefore the prediction of RTG concerning this phenomenon is unambiguous. Note that solution (12.76), which in GR was found by Schwarzschild, corresponds to the case with $\lambda = -1$, and hence the deflection of light in the Sun's gravitational field calculated in the GR framework differs from the RTG result by $4(GM_\odot)^2/r_0^2$. If the accuracy in measuring the deflection of light and radio signals could be raised to a level at which second-order effects came into play, this ambiguity would become experimentally varifiable.

18.4 The Shift of Mercury's Perihelion*

Suppose that a particle of mass $m \neq 0$ is moving along a closed curve around the Sun. To describe this motion we apply formula (18.49). As in Section 18.3 we place the origin of the coordinate system at the Sun's center and superpose the equatorial plane xy with the plane in which the particle moves. Since the trajectory constitutes a closed curve, there are only two values of $r(\varphi)$ at which $dr/d\varphi = 0$. We denote these values by $r_\pm$. Then from (18.48) we have

$$E + \frac{J^2}{W_\pm} - \frac{1}{U_\pm} = 0, \tag{18.69}$$

where $W_\pm = W(r_\pm)$ and $U_\pm = U(r_\pm)$. This can be used to find the integrals of motion, E and J^2, which are given by the following formulas:

$$E = \frac{W_+(\sqrt{W_-} - 2GM_\odot) + W_-(\sqrt{W_+} - 2GM_\odot) - 2GM_\odot\sqrt{W_+}\sqrt{W_-}}{(\sqrt{W_+} + \sqrt{W_-})(\sqrt{W_+} - 2GM_\odot)(\sqrt{W_-} - 2GM_\odot)}, \tag{18.70}$$

$$J^2 = \frac{2GM_\odot W_+ W_-}{(\sqrt{W_+} + \sqrt{W_-})(\sqrt{W_+} - 2GM_\odot)(\sqrt{W_-} - 2GM_\odot)}. \tag{18.71}$$

In deriving these two formulas we allowed for the relationship between U and W (see (18.25)).

The angle $\varphi(r)$ through which the radius vector $\mathbf{r}$ of the particle rotates (this angle is reckoned from the direction specified by $\mathbf{r} = \mathbf{r}_-$) can be calculated according to the formula

$$\varphi(r) = \varphi(r_-) + J \int\limits_{\sqrt{W_-}}^{\sqrt{W(r)}} \frac{d\sqrt{W}}{(\sqrt{W}\,[W^{3/2}(1-E) - J^2(\sqrt{W} - 2GM_\odot) + 2EWGM_\odot])^{1/2}}, \tag{18.72}$$

which emerges as a result of integrating (18.49) with respect to r and allowing for the relationship between $V(r)$ and $W(r)$ (see (18.25)).

Putting $W(r) = W_+$ in (18.72) and allowing for (18.70) and (18.71), we obtain

$$\varphi(r_+) - \varphi(r_-) = \left(\frac{W_+W_-}{\sqrt{W_+}\sqrt{W_-} - 2GM_\odot\sqrt{W_+} - 2GM_\odot\sqrt{W_-}}\right)^{1/2}$$

$$\times \int\limits_{\sqrt{W_-}}^{\sqrt{W_+}} \frac{d\sqrt{W}}{[\sqrt{W}(\sqrt{W} - \sqrt{W_-})(\sqrt{W_+} - \sqrt{W})(\sqrt{W} - \widetilde{W}_0)]^{1/2}}, \tag{18.73}$$

* See Logunov and Loskutov, 1986c, Logunov, Loskutov, and Chugreev, 1986, and Weinberg, 1972.

with

$$\widetilde{W}_0 = \frac{2GM_\odot \sqrt{W_+}\sqrt{W_-}}{\sqrt{W_+}\sqrt{W_-} - 2GM_\odot\sqrt{W_+} - 2GM_\odot\sqrt{W_-}}. \tag{18.74}$$

Integrating in (18.73) yields the following formula for $\varphi(r_+) - \varphi(r_-)$:

$$\varphi(r_+) - \varphi(r_-) = \left(\frac{2\widetilde{W}_0\sqrt{W_+}}{GM_\odot(\sqrt{W_+} - \widetilde{W}_0)}\right)^{1/2} F\left(\frac{\pi}{2}, q\right), \tag{18.75}$$

where

$$q = \left(\frac{\widetilde{W}_0(\sqrt{W_+} - \sqrt{W_-})}{\sqrt{W_-}(\sqrt{W_+} - \widetilde{W}_0)}\right)^{1/2}, \tag{18.76}$$

and

$$F\left(\frac{\pi}{2}, q\right) = \int_0^{\pi/2} (1 - q^2 \sin^2 t)^{-1/2}\, dt$$

is a complete elliptic integral of the first kind.

Formula (18.75) is true for all $W_\pm$ and $\widetilde{W}_0$ satisfying the inequalities

$$\sqrt{W_+} > \sqrt{W_-} > \widetilde{W}_0 > 2GM_\odot, \quad \sqrt{W_-} > 6GM_\odot.$$

It is shown in Appendix 4 that when $\sqrt{W_\pm} \gg 2GM_\odot$, (18.75) leads to the following expansion for $\varphi(r_+) - \varphi(r_-)$, valid to within the second order in $2GM_\odot/\sqrt{W_\pm}$:

$$\varphi(r_+) - \varphi(r_-) \simeq \pi\left[1 + \frac{3GM_\odot}{2}\left(\frac{1}{\sqrt{W_+}} + \frac{1}{\sqrt{W_-}}\right) + \frac{57(GM_\odot)^2}{16}\left(\frac{1}{W_+} + \frac{1}{W_-}\right) + \frac{51}{8}\frac{(GM_\odot)^2}{\sqrt{W_+}\sqrt{W_-}}\right]. \tag{18.77}$$

The variation of angle φ when the particle moves from point $r = r_-$ to point $r = r_+$ must be equal to the variation of angle φ when the particle moves back, from point $r = r_+$ to point $r = r_-$. This means that the total variation of angle φ in the course of a complete rotation is

$$\Delta\varphi = 2\pi + 3\pi GM_\odot\left(\frac{1}{\sqrt{W_+}} + \frac{1}{\sqrt{W_-}}\right) + \frac{57}{8}\pi(GM_\odot)^2\left(\frac{1}{W_+} + \frac{1}{W_-}\right) + \frac{51}{4}\pi\frac{(GM_\odot)^2}{\sqrt{W_+}\sqrt{W_-}}. \tag{18.78}$$

We see that the curve along which the particle moves is not closed. It precesses in the direction of the motion of the particles, and the measure of this precession in the second-order approximation in $GM_\odot/\sqrt{W_\pm}$ is the quantity

$$\delta\varphi \equiv \Delta\varphi - 2\pi = 3\pi GM_\odot\left(\frac{1}{\sqrt{W_+}} + \frac{1}{\sqrt{W_-}}\right) + \frac{57}{8}\pi(GM_\odot)^2\left(\frac{1}{W_+} + \frac{1}{W_-}\right) + \frac{51}{4}\pi\frac{(GM_\odot)^2}{\sqrt{W_+}\sqrt{W_-}}. \tag{18.79}$$

We select the relationship between $W_\pm$ and $r_\pm$ in the form of the one-parameter family

$$W_\pm = [r_\pm + (1+\lambda)\, GM_\odot]^2. \tag{18.80}$$

Then for all values of the adjustable parameter λ satisfying the condition

$$r_\pm \gg GM_\odot \,|\, 1+\lambda \,|$$

we have the following expansion of $\delta\varphi$ in powers of $GM_\odot/r_\pm$ to within second-order terms:

$$\delta\varphi \simeq 3\pi GM_\odot \left(\frac{1}{r_+}+\frac{1}{r_-}\right) + \frac{51\pi}{4}\,\frac{(GM_\odot)^2}{r_+r_-} + 3\pi\,(GM_\odot)^2\left(\frac{11}{8}-\lambda\right)\left(\frac{1}{r_+^2}+\frac{1}{r_-^2}\right). \tag{18.81}$$

The shift given by (18.81) can be expressed in terms of the characteristics p and e of the trajectory:

$$\delta\varphi = \frac{6\pi GM_\odot}{p}\left[1-\frac{(1+\lambda)\,GM_\odot}{p}\,(1+e^2)+\frac{9GM_\odot}{2p}\left(1+\frac{e^2}{18}\right)\right], \tag{18.82}$$

where $2p$ is the *latus rectum* (sometimes p is called the focal parameter) and e the eccentricity linked to r_+ and r_- through the following relationships:

$$p=\frac{2r_+r_-}{r_++r_-},\quad e=\frac{r_+-r_-}{r_++r_-}. \tag{18.83}$$

As (18.81) clearly shows, the ambiguity in the prediction of GR concerning the shift $\delta\varphi$ manifests itself, as in the case with light deflection, in second-order terms in $GM_\odot/r_\pm$, and disappears in RTG since here $\lambda \equiv 0$.

Applying formula (18.81) to the motion of Mercury around the Sun, we arrive in the very first order in $GM_\odot/r_\pm$ at the following value for $\delta\varphi$ (in seconds of arc per century):

$$\delta\varphi = 42.98''/\text{century}.$$

The results of observation yield (see Misner, Thorne, and Wheeler, 1973 (p. 1113), and Will, 1981) $\delta\varphi_{\text{exper}} = (41.1\pm0.9)''/\text{century}$. We see that the present level of experiments in this field is not sufficiently high to study the second-order corrections and, hence, to experimentally determine the value of λ.

Study of the Mercury perihelion shift is further complicated by the fact that a number of other factors, besides post-Newtonians corrections in the equation of motion, affect the perihelion shift. Among these are, say, the gravitational pulls of other planets in the solar system and the deformation of the Sun (the quadrupole moment of the Sun). The only indeterminate factor is the value of the quadrupole moment of the Sun, since the effect of all the other factors can be calculated with sufficient accuracy.

The additional Mercury perihelion shift brought on by the Sun's quadrupole moment J_2 is (in seconds of arc per century)

$$\delta\varphi_{\text{add}} = 1.3 \times 10^5 J_2.$$

Measurements conducted by Dicke and Goldenberg, 1967, on the Sun's apparent oblateness provided a value of the quadrupole moment J_2 equal to $(2.5\pm0.2)\times 10^{-5}$, while later measurements by Hill *et al.*, 1974, (see also Will, 1978, 1979, 1981) yielded the estimate $J_2 < 0.5\times10^{-5}$. A comparison of the perihelion shifts of Mercury and Mars yielded an estimate $J_2 < 3\times10^{-5}$ (see Shapiro *et al.*, 1972a, b).

Thus, the absence of direct measurements of the Sun's quadrupole moment results in a large indeterminacy, which makes it impossible to study the Mercury perihelion shift with good accuracy.

18.5 Time Delay of Radio Signals in the Sun's Gravitational Field (Shapiro's Effect)*

The purpose of the experiment is to measure the time of propagation of a radio signal in the gravitational field of the Sun. Here is how this experiment can be realized. A radar transmitter on the Earth sends a radio wave out to a reflector elsewhere in the solar system (Misner, Thorne, Wheeler, 1973, p. 1106). The reflected wave is then received on the Earth. The round-trip travel time is measured by a clock on the Earth. This quantity is compared to the round-trip travel time in the absence of the Sun's gravitational field. In this way the time delay of radio signals in the gravitational field of the Sun is determined.

To calculate theoretically the time delay of radio signals in the Sun's gravitational field let us turn to Eq. (18.46). Since for radio signals $E = 0$, Eq. (18.46) yields

$$\frac{dt}{dr} = \frac{[V(r)]^{1/2}}{U(r)} \left[\frac{1}{U(r)} - \frac{J^2}{W(r)}\right]^{-1/2}. \tag{18.84}$$

We place the origin of the coordinate system at the Sun's center and assume that the radio signal propagates in the equatorial plane xy. Let r_0 be the distance from the origin to the point where the path of the radio signal comes closest to the source of gravitational field. Then $J^2 = W_0/U_0$, where $W_0 = W(r_0)$ and $U_0 = U(r_0)$. It is obvious that in experiments of this type we must consider only paths of radio signals for which $r_0 \geqslant R_\odot$. Integrating (18.84) with respect to r and allowing for the relationships between $U(r)$, $V(r)$, and $W(r)$ (see (18.25)), we get

$$t(r_0, r) = \int\limits_{\sqrt{W_0}}^{\sqrt{W(r)}} \frac{(\sqrt{W})^{5/2}\, d\sqrt{W}}{(\sqrt{W} - 2GM_\odot)\,[(\sqrt{W} - \sqrt{W_0})(\sqrt{W} - W_1)(\sqrt{W} - W_2)]^{1/2}}, \tag{18.86}$$

where W_1 and W_2 are given in (18.57) and (18.58), respectively. This formula gives the time that it takes the radio signal to propagate from point r_0 to point r.

For $\sqrt{W_0} > 3GM_\odot$ we have $\sqrt{W_0} > W_2 > 0 > W_1$, with the result that the integral (18.86) can be represented thus:

$$t(r_0, r) = (2GM_\odot)^3 I_3(r) + (2GM_\odot)^2 I_2(r) + (2GM_\odot) I_1(r) + I_0(r), \tag{18.87}$$

where

$$I_3(r) = \frac{2}{(\sqrt{W_0} - 2GM_\odot)(W_2 - 2GM_\odot)[\sqrt{W_0}(W_2 - W_1)]^{1/2}}$$
$$\times \left[(\sqrt{W_0} - 2GM_\odot) F(\nu, q) - (\sqrt{W_0} - W_2)\, \Pi\left(\nu, \frac{(\sqrt{W_0} - W_1)(W_2 - 2GM_\odot)}{(W_2 - W_1)(\sqrt{W_0} - 2GM_\odot)}, q\right)\right], \tag{18.88}$$

$$I_2(r) = \frac{2}{[\sqrt{W_0}(W_2 - W_1)]^{1/2}} F(\nu, q), \tag{18.89}$$

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* See Logunov and Loskutov, 1985a, 1985b, 1986a, 1986b, 1986c, Logunov, Loskutov, and Chugreev, 1986, and Shapiro, 1964, 1979.

$$I_1(r) = \frac{2}{[\sqrt{W_0}(W_2-W_1)]^{1/2}} \times \left[W_2 F(\nu,\, q) + (\sqrt{W_0} - W_2)\,\Pi\left(\nu,\, \frac{\sqrt{W_0}-W_1}{W_2-W_1},\, q\right)\right], \tag{18.90}$$

$$I_0(r) = -\frac{\partial}{\partial P}\left\{\frac{2W_2(\sqrt{W_0})^{1/2}}{(PW_2-1)(P\sqrt{W_0}-1)\sqrt{W_2-W_1}} \times \left[\frac{P\sqrt{W_0}-1}{\sqrt{W_0}} F(\nu,\, q) + \frac{W_2-\sqrt{W_0}}{W_2\sqrt{W_0}}\,\Pi\left(\nu,\, q^2\frac{\sqrt{W_0}}{W_2}\;\frac{PW_2-1}{P\sqrt{W_0}-1},\, q\right)\right]\right\}_{P=0}, \tag{18.91}$$

where q is given by (18.62),

$$\nu = \sin^{-1}\left[\frac{(W_2-W_1)(\sqrt{W(r)}-\sqrt{W_0})}{(\sqrt{W_0}-W_1)(\sqrt{W(r)}-W_2)}\right]^{1/2}, \tag{18.92}$$

$F(\nu, q)$ is an elliptic integral of the first kind, and $\Pi(\nu, \sigma, q)$ is an elliptic integral of the third kind:

$$\Pi(\nu,\ \sigma,\ q) = \int_0^\nu \frac{d\alpha}{(1+\sigma\sin^2\alpha)\sqrt{1-q^2\sin^2\alpha}}.$$

Suppose that in the selected reference frame the Earth has coordinates $(r_e,\ \varphi_e)$ and the reflector, coordinates $(r_\mu,\ \varphi_\mu)$. In Shapiro's experiments the reflector was Mercury in superior conjunction $(\varphi_e - \varphi_\mu \simeq \pi)$.

On the basis of (18.87)-(18.91) we arrive at the following formula for the time it takes the radio signal to propagate from the Earth to Mercury in the Sun's gravitational field:

$$t(r_e,\ r_\mu) = (2GM_\odot)^3[I_3(r_e) + I_3(r_\mu)] + (2GM_\odot)^2[I_2(r_e) + I_2(r_\mu)] + 2GM_\odot[I_1(r_e) + I_1(r_\mu)] + [I_0(r_e) + I_0(r_\mu)]. \tag{18.93}$$

If we select the paths followed by radio signals for which $r_0 \simeq R_\odot$ and allow for the fact that $\sqrt{W_0} \gg 2GM_\odot$, while in the given experiment $r_e \gg r_0$ and $r_\mu \gg r_0$ and, hence, $W_e \equiv W(r_e) \gg W_0$ and $W_\mu \equiv W(r_\mu) \gg W_0$, we find from (18.93) that to within the first order in $GM_\odot$ the following formula holds true for $t(r_e, r_\mu)$ (see Appendix 4):

$$t(r_e,\ r_\mu) \simeq (W_e - W_0)^{1/2} + (W_\mu - W_0)^{1/2} + GM_\odot\left[2\ln\frac{\sqrt{W_\mu}+\sqrt{W_\mu-W_0}}{\sqrt{W_e}-\sqrt{W_e-W_0}} + \left(\frac{\sqrt{W_\mu}-\sqrt{W_0}}{\sqrt{W_\mu}+\sqrt{W_0}}\right)^{1/2} + \left(\frac{\sqrt{W_e}-\sqrt{W_0}}{\sqrt{W_e}+\sqrt{W_0}}\right)^{1/2}\right]. \tag{18.95}$$

For further analysis we need the equation describing the trajectory of the light signal $W = W(\varphi)$. This equation can be found from (18.49). If we put $E = 0$ in Eq. (18.49), allow for (18.25), and integrate, we obtain in the first order in $GM_\odot$ the following:

$$\frac{1}{\sqrt{W}} = \frac{1+e\cos\chi}{P}, \tag{18.96a}$$

where

$$\frac{1}{P}=\frac{GM_\odot}{W_0},\quad \frac{e}{P}=\frac{1}{\sqrt{W_0}}\left(1-\frac{GM_\odot}{\sqrt{W_0}}\right),\quad \chi=\psi-\frac{GM_\odot}{\sqrt{W_0}}\sin\psi. \tag{18.96b}$$

In (18.96b), $\psi=0$ corresponds to the direction to the pericenter of the light signal. The angles reckoned from this direction for Earth and Mercury will be denoted by ψ_e and ψ_μ, respectively. Since in real experiments $\sqrt{W_e}\cos^2\psi_e \gg 8GM_\odot$ and $\sqrt{W_\mu}\cos^2\psi_\mu \gg 8GM_\odot$, Eqs. (18.96a) and (18.96b) yield, in the first order in $GM_\odot$, the following:

$$\begin{aligned}\sqrt{W_0} &\simeq \sqrt{W_e}\cos\psi_e+GM_\odot\frac{1+\sin^2\psi_e}{\cos\psi_e}-GM_\odot\\ &=\sqrt{W_\mu}\cos\psi_\mu+GM_\odot\frac{1+\sin^2\psi_\mu}{\cos\psi_\mu}-GM_\odot.\end{aligned} \tag{18.97a}$$

Substituting (18.97a) into (18.96a) and retaining terms up to the first order in $GM_\odot$, we find that

$$\begin{aligned}t(e,\mu) &= \sqrt{W_e}\sin\psi_e+\sqrt{W_\mu}\sin\psi_\mu-GM_\odot(\sin\psi_e+\sin\psi_\mu)\\ &\quad+GM_\odot\ln\frac{(1+\sin\psi_e)(1+\sin\psi_\mu)}{(1-\sin\psi_e)(1-\sin\psi_\mu)}.\end{aligned} \tag{18.97b}$$

To calculate the variation in the flow of time caused by the Sun's gravitational field, we must subtract from (18.97b) the value of the time it takes the radio signal to travel from point r_e to point r_μ in the absence of the Sun's gravitational field.

Sending G to zero in Eq. (18.97b) and bearing in mind that $W_{e,\mu}$ and $\psi_{e,\mu}$ depend on G, we get

$$t_0=\lim_{G\to 0} t(e,\mu)=\rho_e\sin\varphi_e+\rho_\mu\sin\varphi_\mu, \tag{18.97c}$$

where

$$\rho_{e,\mu}=\lim_{G\to 0}\sqrt{W_{e,\mu}},\quad \varphi_{e,\mu}=\lim_{G\to 0}\psi_{e,\mu}. \tag{18.97d}$$

Let us establish the meaning of the limiting quantities $\rho_{e,\mu}$ and $\varphi_{e,\mu}$. Since the line element with metric coefficients (18.25) tends to

$$ds^2=dt^2-d\rho^2-\rho^2(d\theta^2+\sin^2\theta\, d\varphi^2)$$

as $G\to 0$ and $\lim\limits_{G\to 0}\sqrt{W}=\rho$, and the latter coincides with the line element of the Minkowski space-time in standard coordinates (t, r, θ, φ), we conclude that $\rho\equiv r$ and, hence, $\rho_{e,\mu}=r_{e,\mu}$.

Putting $G=0$ in Eqs. (18.96a) and (18.96b), we find that

$$\frac{1}{r}=\frac{1}{\rho_0}\cos\varphi, \tag{18.98a}$$

where

$$\rho_0=\lim_{G\to 0}\sqrt{W_0},\quad \varphi=\lim_{G\to 0}\psi.$$

Equation (18.98a) implies

$$r_e\cos\varphi_e=r_\mu\cos\varphi_\mu, \tag{18.98b}$$

where $\varphi_{e,\mu}$ are the values of the polar angle for the Earth and Mercury, respectively.

Subtracting (18.97c) from (18.97b), we arrive at the formula for the time delay of the radio signal in the Sun's gravitational field:

$$\Delta t = t\,(e,\ \mu) - t_0. \tag{18.98c}$$

But since the function $W\,(r,\ GM_{\odot})$ is not determined by the Hilbert-Einstein equations, the dependence of $W_{e,\mu}$ on $r_{e,\mu}$ and $GM_{\odot}$ is unknown within the framework of GR and, therefore, the right-hand side of (18.98c) cannot be calculated unambiguously. Let us illustrate this ambiguity with the example of the one-parameter family of functions

$$W\,(r,\ GM_{\odot}) = [r + (1 + \lambda)\ GM_{\odot}]^2, \tag{18.98d}$$

where λ is an arbitrary finite parameter. It is easy to see that for such a choice of $W\,(r,\ GM_{\odot})$ the solutions (18.25) at $G = 0$ are transformed into the metric coefficients γ_{mn} of the pseudo-Euclidean space-time in spherical coordinates.

Bearing in mind the spherical symmetry of the problem, in view of which $\psi_e + \psi_\mu = \varphi_e + \varphi_\mu$, we introduce the notation

$$\psi_e = \varphi_e - \delta,\quad \psi_\mu = \varphi_\mu + \delta. \tag{18.98e}$$

Then, combining (18.97a), (18.98b), and (18.98d), we arrive at the following expression for δ in the first order in $GM_{\odot}$:

$$\delta = \frac{GM_{\odot}\,(\cos\varphi_e - \cos\varphi_\mu)\,(2 - \lambda\cos\varphi_e\cos\varphi_\mu)}{(r_e\sin\varphi_e + r_\mu\sin\varphi_\mu)\cos\varphi_e\cos\varphi_\mu}\,. \tag{18.98f}$$

Substituting (18.98d), (18.98e), and (18.98f) into (18.97b) yields the following expression for the time the radio signal takes to propagate in the Sun's gravitational field:

$$t\,(e,\ \mu) = r_e\sin\varphi_e + r_\mu\sin\varphi_\mu + \lambda GM_{\odot}\,(\sin\varphi_e + \sin\varphi_\mu) + GM_{\odot}\ln\frac{(1+\sin\varphi_e)\,(1+\sin\varphi_\mu)}{(1-\sin\varphi_e)\,(1-\sin\varphi_\mu)}\,. \tag{18.99a}$$

The same quantity can be expressed in terms of radial distances, which are determined, according to metric (18.25), via (18.98d). Since by definition

$$l_{e,\,\mu} = \int\limits_{r_0}^{r_{e,\,\mu}} \left[\frac{r + (1+\lambda)\,GM_{\odot}}{r - (1-\lambda)\,GM_{\odot}}\right]^{1/2} dr,$$

with r_0 the radial arithmetization number corresponding to points on the surface of the Sun, we integrate and arrive, in the first order in $GM_{\odot}$, at the following:

$$l_{e,\mu} \simeq r_{e,\mu} - r_0 + GM_{\odot}\ln\,(r_{e,\mu}/r_0), \tag{18.99b}$$

which yields

$$l_e - l_\mu \simeq r_e - r_\mu + GM_{\odot}\ \ln\,(r_e/r_\mu).$$

In the first order in $GM_{\odot}$ this equation allows for an inversion of the type

$$r_e - r_\mu \simeq l_e - l_\mu - GM_{\odot}\ln\,(l_e/l_\mu) \equiv a,$$

and, therefore, allowing for (18.98b), we easily find that

$$r_e = a\,\frac{\cos\varphi_\mu}{\cos\varphi_\mu - \cos\varphi_e}\,,\quad r_\mu = \frac{\cos\varphi_e}{\cos\varphi_\mu - \cos\varphi_e}\,.$$

Substituting these expressions into (18.99a), we find that

$$t(e,\ \mu) = a\frac{\sin(\varphi_e+\varphi_\mu)}{\cos\varphi_\mu-\cos\varphi_e} + \lambda GM_\odot(\sin\varphi_e+\sin\varphi_\mu) + GM_\odot \ln\frac{(1+\sin\varphi_e)(1+\sin\varphi_\mu)}{(1-\sin\varphi_e)(1-\sin\varphi_\mu)}. \tag{18.99c}$$

Formulas (18.99a) and (18.99c) demonstrate that already in the first order in $GM_\odot$ the magnitude of the time it takes the radio signal to travel in the Sun's gravitational field cannot be found unambiguously in GR since the Hilbert-Einstein equations do not fix the value of parameter λ. Hence, the statement made by Ichinose and Kaminaga, 1987, that the predictions of GR are unambiguous are simply erroneous. Note that the above-mentioned ambiguity in the prediction of GR was remarked on earlier (e.g., see Brumberg, 1972), but no meaningful conclusions were drawn.

Substituting (18.99a) and (18.97c) into (18.98c), we arrive at the following formula for the time delay of radio signals in the gravitational field of the Sun:

$$\Delta t = \lambda GM_\odot(\sin\varphi_e+\sin\varphi_\mu) + GM_\odot\ln\frac{(1+\sin\varphi_e)(1+\sin\varphi_\mu)}{(1-\sin\varphi_e)(1-\sin\varphi_\mu)}. \tag{18.100a}$$

The presence in (18.100a) of the indeterminate parameter λ makes it impossible to find a definite value of Δt remaining with the framework of GR.

In RTG the value of parameter λ is well-defined (it is equal to zero), and therefore for the time delay of radio signals in the Sun's gravitational field we have an unambiguous formula:

$$\Delta t = GM_\odot\ln\frac{(1+\sin\varphi_e)(1+\sin\varphi_\mu)}{(1-\sin\varphi_e)(1-\sin\varphi_\mu)}. \tag{18.100b}$$

Note that if the relative frequency shift of a radio signal propagating in the Sun's field is calculated on the basis of (18.98d), then in the first order in $GM_\odot$ we obtain (see Logunov and Loskutov, 1987b)

$$\delta_{e,\mu} = \frac{\Delta\omega}{\omega}\bigg|_{e,\mu} \simeq GM_\odot\left(\frac{1}{r_0}-\frac{1}{r_{e,\mu}}\right). \tag{18.101a}$$

We see that $\delta_{e,\mu}$ is independent of the parameter λ. We will derive (18.101a) when considering the ambiguities of the predictions of GR.

Here is another example. It is often stated in the literature that if the time delay of a radio signal in a gravitational field, $\Delta t\,(r_e,\ r_\mu)$, is expressed in terms of the periods of revolution of the Earth, T_e, Mercury, T_μ, and a test body orbiting the Sun along an orbit of radius $r = r_0 \simeq R_\odot$, T_0, then $\Delta t\,(r_e,\ r_\mu)$ is independent of the choice of (12.72) or (12.76) as solution. We will now prove this statement to be erroneous. For the sake of simplicity we assume that the above noted objects revolve about the source of gravitational field (the Sun) along circular orbits. Then, under the above-stated arithmetization of space with line elements defined in (12.72) and (12.76) we arrive in the first order in $GM_\odot$ at the following results (see Logunov and Loskutov, 1986a, 1987a):

$$T_{0,e,\mu}(\lambda=0) = 2\pi\frac{r_{0,e,\mu}^{3/2}}{\sqrt{GM_\odot}}\left(1+\frac{3}{2}\ \frac{GM_\odot}{r_{0,e,\mu}}\right), \tag{18.101b}$$

$$T_{0,e,\mu}(\lambda=-1) = 2\pi\frac{r_{0,e,\mu}^{3/2}}{\sqrt{GM_\odot}}. \tag{18.101c}$$

We see that according to GR the periods of revolution of the objects in the orbits prove to be, in the variables (t, r, θ, φ), different for different metrics. If in (18.101b) and (18.101c) we pass from numbers r to observables $l_{e,\mu}$ and $\delta_{e,\mu}$, in terms of these measurable variables the ambiguities in the theoretical values of $T_{0,e,\mu}$ remain. The difference in the periods of revolution corresponding to metrics (12.72) and (12.76) can be explained here by the fact that although the physical radial distances to the orbit in different metrics coincide (in the first order in $GM_\odot$), the velocity of an object differs from metric to metric (Logunov and Loskutov, 1986a).

If we now express the time of propagation $t\ (\lambda = 0)$ (or $t\ (\lambda = -1)$) in terms of the periods of revolution $T_{0,e,\mu}\ (\lambda = 0)$ (or $T_{0,e,\mu}\ (\lambda = -1)$) and introduce, for the sake of simplicity, the notation

$$L_{0,\,e,\,\mu}=\left(\frac{\sqrt{GM_\odot}}{2\pi}T_{0,\,e,\,\mu}\right)^{2/3},$$

then for both metrics, (12.72) and (12.76), the t vs. T relation is the same:

$$t=\sqrt{L_\mu^2-L_0^2}+\sqrt{L_e^2-L_0^2}+GM_\odot\left\{2\ln\frac{L_\mu+\sqrt{L_\mu^2-L_0^2}}{L_e-\sqrt{L_e^2-L_0^2}}+\left(\frac{L_\mu-L_0}{L_\mu+L_0}\right)^{1/2}+\left(\frac{L_e-L_0}{L_e+L_0}\right)^{1/2}\right\}. \quad (18.102a)$$

To determine the effect of gravitational time delay proper, which actually constitutes the goal of such studies, one must also isolate the time t_0 necessary for the radio signal to cover the distance from the Earth to Mercury in the absence of the Sun's gravitational field. This means that we must additionally calculate the time t_0 it would take the signal to propagate from (e) to (μ) in the flat metric γ_{ik} and express this time, via (18.101b) and (18.101c), in terms of the revolution periods T. In the initial arithmetization of space,

$$t_0=\sqrt{r_\mu^2-r_\perp^2}+\sqrt{r_e^2-r_\perp^2}.$$

Correspondingly, with metrics (12.72) and (12.76), that is, with (18.101b) and (18.101c), we have*

$$t_0\,(\lambda=0)=\sqrt{L_\mu^2-L_\perp^2}+\sqrt{L_e^2-L_\perp^2}-GM_\odot\left[\left(\frac{L_\mu-L_\perp}{L_\mu+L_\perp}\right)^{1/2}+\left(\frac{L_e-L_\perp}{L_e+L_\perp}\right)^{1/2}\right], \quad (18.102b)$$

$$t_0\,(\lambda=-1)=\sqrt{L_\mu^2-L_\perp^2}+\sqrt{L_e^2-L_\perp^2}. \quad (18.102c)$$

Thus, the time delay due to gravitation, $\Delta t = t - t_0$, is defined differently in metrics (12.72) and (12.76). For $GM_\odot$, $L_\perp$, and L_0 much smaller than $L_{e,\,\mu}$ we obtain

$$\Delta t\,(\lambda=0)=2GM_\odot\ln\frac{L_e+L_\mu+t_0\,(\lambda=0)}{L_e+L_\mu-t_0\,(\lambda=0)}, \quad (18.102d)$$

$$\Delta t\,(\lambda=-1)=2GM_\odot\ln\frac{L_e+L_\mu+t_0\,(\lambda=0)}{L_e+L_\mu-t_0\,(\lambda=0)}-2GM_\odot. \quad (18.102e)$$

* Here we introduce $L_\perp$ quite formally; however, when the pericenter of the radio signal's path is far from the Sun's surface, this quantity can be realized via an auxiliary test body orbiting the Sun along a circular orbit with $r = r_\perp$.

The calculation of t can be carried out if from the start we pass from coordinates (t, r, θ, φ) to variables $(t, \rho, \theta, \varphi)$, with $\rho = \sqrt{W(r)}$, in terms of which

$$ds^2 = \left(1 - \frac{2GM_\odot}{\rho}\right) dt^2 - \left(1 - \frac{2GM_\odot}{\rho}\right)^{-1} d\rho^2 - \rho^2 (d\theta^2 + \sin^2\theta\, d\varphi^2). \quad (18.102f)$$

On the basis of expression for the line element we can obtain, in the first order in $GM_\odot$, the following formula for t:

$$t = \sqrt{\rho_\mu^2 - \rho_0^2} + \sqrt{\rho_e^2 - \rho_0^2} + GM_\odot \left\{ 2 \ln \frac{\rho_\mu + \sqrt{\rho_\mu^2 - \rho_0^2}}{\rho_e - \sqrt{\rho_e^2 - \rho_0^2}} + \left(\frac{\rho_\mu - \rho_0}{\rho_\mu + \rho_0}\right)^{1/2} + \left(\frac{\rho_e - \rho_0}{\rho_e + \rho_0}\right)^{1/2} \right\},$$

and for the periods $T_{0,e,\mu}$ we have

$$T_{0,e,\mu} = 2\pi \frac{\rho_{0,e,\mu}^{3/2}}{\sqrt{GM_\odot}}.$$

Although the link between t and the experimentally measurable $T_{0,e,\mu}$ that follows from the above formulas is unambiguous, the time delay Δt will again be different for different metrics g_{ik}, since, introducing the metric tensor γ_{ik} of the flat space-time in the arithmetizations r in order to calculate t_0, we again arrive at (18.102d) and (18.102e) if we take ρ to be equal to $r + GM_\odot$ and r, respectively; and the introduction of a flat metric $\gamma_{ik}(\rho)$ leads to an ambiguity because of the indeterminacy in the value of ρ_0: say, for $\rho_0 = GM_\odot$ we have

$$t_0 = \sqrt{\rho_\mu^2 - \rho_\perp^2} + \sqrt{\rho_e^2 - \rho_\perp^2} - GM_\odot \left[\left(\frac{\rho_\mu - \rho_\perp}{\rho_\mu + \rho_\perp}\right)^{1/2} + \left(\frac{\rho_e - \rho_\perp}{\rho_e + \rho_\perp}\right)^{1/2} \right],$$

and for $\rho_0 = 0$ we have

$$t_0 = \sqrt{\rho_\mu^2 - \rho_\perp^2} + \sqrt{\rho_e^2 - \rho_\perp^2},$$

so that for $GM_\odot$, $\rho_\perp$, and ρ_0 much smaller than $\rho_{e,\mu}$ we again arrive at formulas (18.102d) and (18.102e) for Δt. Note that the value of Δt calculated on the basis of (18.102e) does not agree with the experimentally observed value (see Shapiro, 1979).

Finally, we note that in terms of the variables $(t, \rho, \theta, \varphi)$ solution (18.102f) is not the only solution to the Hilbert-Einstein equations. Indeed, as already noted in Chapter 12, the function $\rho = \sqrt{W(r)}$ can always be taken as one of the variables in whose terms the Hilbert-Einstein equation is written, with the result that (18.25) will also be solutions to this equation, the only difference being that in the latter solutions ρ plays the role of r and the function $W(\rho)$ plays the role of $W(r)$, that is, ambiguities in t and Δt emerge because of the arbitrariness in the choice of $W(\rho)$, as was the case earlier in the initial arithmetization in terms of (t, r, θ, φ).

18.6 Period of Revolution of a Test Body in Orbit*

To determine the period of revolution of a test body in orbit we will start with Eq. (18.46) and the equation for the trajectory

$$r = \frac{p}{1 + e \cos q\varphi}, \quad (18.103)$$

* See Logunov and Loskutov, 1986b, 1986c, and Logunov, Loskutov, and Chugreev, 1986.

where

$$p=\frac{2r_+r_-}{r_++r_-}, \quad e=\frac{r_+-r_-}{r_++r_-}, \tag{18.104a}$$

$$q\simeq 1-\frac{3Gm}{p}, \tag{18.104b}$$

r_- is the minimal distance from the center of the source of gravitational field to the path of the test body, and r_+ is the maximal distance.

Since the motion is finite, for E and J^2 we have (18.70) and (18.71), respectively. From (18.46) it readily follows that

$$\frac{dt}{dr}=\frac{\sqrt{WV}}{\sqrt{U}\,[W-J^2U-EUW]^{1/2}}. \tag{18.105}$$

Placing the origin of the coordinate system at the center of the source of gravitational field, superposing the equatorial plane xy with the plane in which the test body moves, and integrating (18.105), we arrive at the following formula for the period of revolution of the test body:

$$T=2\int\limits_{r_-}^{r_+}\frac{dr\,\sqrt{WV}}{\sqrt{U}\,[W-J^2U-EUW]^{1/2}}-\int\limits_{r_-}^{r_1}\frac{dr\,\sqrt{WV}}{\sqrt{U}\,[W-J^2U-EUW]^{1/2}}. \tag{18.106}$$

The origin of the last integral in (18.106) can easily be traced if we note that under a complete revolution, that is, when φ varies from 0 to 2π, the test body does not, according to (18.103), return to the initial point but is found at the point

$$r_1=r(\varphi=2\pi)=\frac{p}{1+e\cos 2\pi q}. \tag{18.107}$$

Allowing in (18.106) for the relationship between functions U and V and the function $W(r)$ (see (18.25)), we obtain

$$T=\Big(2\int\limits_{\sqrt{W_-}}^{\sqrt{W_+}}-\int\limits_{\sqrt{W_-}}^{\sqrt{W_1}}\Big)\frac{d\sqrt{W}\,(\sqrt{W})^{5/2}}{(\sqrt{W}-2Gm)\,[(1-E)\,W^{3/2}-J^2\,(\sqrt{W}-2Gm)+2GmEW]^{1/2}}, \tag{18.108}$$

where $W_\pm=W(r_\pm)$, $W_1=W(r_1)$, and m is the mass of the source of gravitational field. Substituting into (18.108) the expressions (18.70) and (18.71), we find that

$$T=\Big[\frac{(\sqrt{W_+}+\sqrt{W_-})\,(\sqrt{W_+}-2Gm)\,(\sqrt{W_-}-2Gm)}{2Gm\,(\sqrt{W_+}\sqrt{W_-}-2Gm\sqrt{W_+}-2Gm\sqrt{W_-})}\Big]^{1/2}\Big(2\int\limits_{\sqrt{W_-}}^{\sqrt{W_+}}-\int\limits_{\sqrt{W_-}}^{\sqrt{W_1}}\Big)$$
$$\times\frac{d\sqrt{W}\,(\sqrt{W})^{5/2}}{(\sqrt{W}-2Gm)\,[(\sqrt{W_+}-\sqrt{W})\,(\sqrt{W}-\sqrt{W_-})\,(\sqrt{W}-\widetilde{W}_0)]^{1/2}}, \tag{18.109}$$

with

$$\widetilde{W}_0\equiv\frac{2Gm\sqrt{W_+}\sqrt{W_-}}{\sqrt{W_+}\sqrt{W_-}-2Gm\sqrt{W_+}-2Gm\sqrt{W_-}}. \tag{18.110}$$

For $\sqrt{W_-}\gg 2Gm$ we have $\widetilde{W}_0<\sqrt{W_-}$, with the result that T can be represented as follows:

$$T=\Big[\frac{(\sqrt{W_+}+\sqrt{W_-})\,(\sqrt{W_+}-2Gm)\,(\sqrt{W_-}-2Gm)}{2Gm\,(\sqrt{W_+}\sqrt{W_-}-2Gm\sqrt{W_+}-2Gm\sqrt{W_-})}\Big]^{1/2}$$
$$\times\{(2Gm)^3\,[2\widetilde{I}_3(r_+,\ r_-)-\widetilde{I}_3(r_1,\ r_-)]+(2Gm)^2\,[2\widetilde{I}_2(r_+,\ r_-)-\widetilde{I}_2(r_1,\ r_-)]$$
$$+2Gm\,[2\widetilde{I}_1(r_+,\ r_-)-\widetilde{I}_1(r_1,\ r_-)]+[2\widetilde{I}_0(r_+,\ r_-)-\widetilde{I}_0(r_1,\ r_-)]\}, \tag{18.111}$$

where

$$\widetilde{I}_3(r_+,\ r_-)=\frac{2}{(\sqrt{W}-2Gm)\,(2Gm-\widetilde{W}_0)\,[\sqrt{W_-}\,(\sqrt{W_+}-\widetilde{W}_0)]^{1/2}}$$
$$\times\left[(\sqrt{W_-}-\widetilde{W}_0)\,\Pi\left(\frac{\pi}{2},\ \frac{(\sqrt{W_+}-\sqrt{W_-})\,(2Gm-\widetilde{W}_0)}{(\sqrt{W_+}-\widetilde{W}_0)\,(2Gm-\sqrt{W_-})},\ q\right)\right.$$
$$\left.+(2Gm-\sqrt{W_-})\,F(\pi/2,\ q)\right], \tag{18.112a}$$

$$\widetilde{I}_2(r_+,\ r_-)=\frac{2}{[(\sqrt{W_-}\,(\sqrt{W_+}-\widetilde{W}_0)]^{1/2}}\,F(\pi/2,\ q), \tag{18.112b}$$

$$\widetilde{I}_1(r_+,\ r_-)=\frac{2}{[\sqrt{W_-}\,(\sqrt{W_+}-\widetilde{W}_0)]^{1/2}}$$
$$\times\left[(\sqrt{W_-}-\widetilde{W}_0)\,\Pi\left(\frac{\pi}{2},\ \frac{\sqrt{W_+}-\sqrt{W_-}}{\sqrt{W_-}-\widetilde{W}_0},\ q\right)+\widetilde{W}_0F(\pi/2,\ q)\right], \tag{18.112c}$$

$$\widetilde{I}_0(r_+,\ r_-)=-\frac{\partial}{\partial x}\left\{\frac{1}{(x\widetilde{W}_0-1)\,(x\sqrt{W_-}-1)\,(\sqrt{W_+}-\widetilde{W}_0)^{1/2}}\right.$$
$$\times\left[(\widetilde{W}_0-\sqrt{W_-})\,\Pi\left(\frac{\pi}{2},\ \frac{(\sqrt{W_+}-\sqrt{W_-})\,(x\widetilde{W}_0-1)}{(\sqrt{W_+}-\widetilde{W}_0)\,(x\sqrt{W_-}-1)},\ \sigma\right)\right.$$
$$\left.\left.+\widetilde{W}_0\,(x\sqrt{W_-}-1)\,F(\pi/2,\ \sigma)\right]\right\}_{x=0}, \tag{18.112d}$$

$$\widetilde{I}_3(r_1,\ r_-)=\frac{2}{(\sqrt{W_-}-2Gm)\,(2Gm-\widetilde{W}_0)\,[\sqrt{W_-}\,(\sqrt{W_+}-\widetilde{W}_0)]^{1/2}}$$
$$\times\left[(\sqrt{W_-}-\widetilde{W}_0)\,\Pi\left(\chi,\ \frac{(\sqrt{W_+}-\sqrt{W_-})\,(2Gm-\widetilde{W}_0)}{(\sqrt{W_+}-\widetilde{W}_0)\,(2Gm-\sqrt{W_-})},\ q\right)\right.$$
$$\left.+(2Gm-\sqrt{W_-})\,F(\chi,\ q)\right], \tag{18.113a}$$

$$\widetilde{I}_2(r_1,\ r_-)=\frac{2}{[\sqrt{W_-}\,(\sqrt{W_+}-\widetilde{W}_0)]^{1/2}}\,F(\chi,\ q), \tag{18.113b}$$

$$\widetilde{I}_1(r_1,\ r_-)=\frac{2}{[\sqrt{W_-}\,(\sqrt{W_+}-\widetilde{W}_0)]^{1/2}}$$
$$\times\left[(\sqrt{W_-}-\widetilde{W}_0)\,\Pi\left(\chi,\ \frac{\sqrt{W_+}-\sqrt{W_-}}{\sqrt{W_+}-\widetilde{W}_0},\ q\right)+\widetilde{W}_0F(\chi,\ q)\right], \tag{18.113c}$$

$$\widetilde{I}_0(r_1,\ r_-)=-\frac{\partial}{\partial x}\left\{\frac{1}{(x\widetilde{W}_0-1)\,(x\sqrt{W_-}-1)\,(\sqrt{W_+}-\widetilde{W}_0)^{1/2}}\right.$$
$$\times\left[(\widetilde{W}_0-\sqrt{W_-})\,\Pi\left(\chi,\ \frac{(\sqrt{W_+}-\sqrt{W_-})\,(x\widetilde{W}_0-1)}{(\sqrt{W_+}-\widetilde{W}_0)\,(x\sqrt{W_-}-1)},\ \sigma\right)\right.$$
$$\left.\left.+\widetilde{W}_0\,(x\sqrt{W_-}-1)\,F(\chi,\ \sigma)\right]\right\}_{x=0}. \tag{18.113d}$$

Here

$$q=\left[\frac{(\sqrt{W_+}-\sqrt{W_-})\,\widetilde{W}_0}{(\sqrt{W_+}-\widetilde{W}_0)\,\sqrt{W_-}}\right]^{1/2}, \tag{18.114a}$$

$$\sigma=\left(\frac{\sqrt{W_+}-\sqrt{W_-}}{\sqrt{W_+}-\widetilde{W}_0}\right)^{1/2}, \tag{18.114b}$$

$$\chi=\sin^{-1}\left[\frac{(\sqrt{W_+}-\widetilde{W}_0)(\sqrt{W_1}-\sqrt{W_-})}{(\sqrt{W_+}-\sqrt{W_-})(\sqrt{W_1}-\widetilde{W}_0)}\right]^{1/2}. \tag{18.114c}$$

If the path of motion of the test body is such that $r_\pm \gg Gm$ and $r_1 \gg Gm$ and, hence, $\sqrt{W_\pm} \gg Gm$ and $\sqrt{W_1} \gg Gm$, then (18.111) yields the following formula for the period of revolution (see Appendix 4):

$$T\simeq\pi\frac{(\sqrt{W_+}+\sqrt{W_-})^{3/2}}{\sqrt{2Gm}}\left\{1+\frac{6Gm}{\sqrt{W_+}+\sqrt{W_-}}+\frac{1}{\pi}\frac{[(\sqrt{W_+}-\sqrt{W_1})(\sqrt{W_1}-\sqrt{W_-})]^{1/2}}{\sqrt{W_+}+\sqrt{W_-}}\right.$$
$$\left.-\frac{1}{2\pi}\left[\frac{\pi}{2}-\sin^{-1}\left(1-2\frac{\sqrt{W_1}-\sqrt{W_-}}{\sqrt{W_+}-\sqrt{W_-}}\right)\right]\right\}. \tag{18.115}$$

Let us now take the one-parameter family $\sqrt{W(r)} = r + (1 + \lambda)\, Gm$ as the solution to the Hilbert-Einstein equations. In this case, obviously,

$$\sqrt{W_\pm}=r_\pm+(1+\lambda)\,Gm \tag{18.116a}$$

and

$$\sqrt{W_1}=r_1+(1+\lambda)\,Gm$$
$$\simeq r_-+(1+\lambda)\,Gm+\frac{r_+-r_-}{4}\left(\frac{r_-}{r_+}\right)\left[\frac{3\pi Gm\,(r_++r_-)}{r_+r_-}\right]^2, \tag{18.116b}$$

where we have allowed for the approximate expression for r_1 to within terms of the order of $(Gm)^2$, an expression that can be obtained from (18.107), (18.104a), and (18.104b). Then for all values of parameter λ satisfying the condition $r_\pm \gg |1 + \lambda|\, Gm$ we obtain, if we combine (18.115) with (18.116a) and (18.116b), the following formula for the period of revolution:

$$T\simeq\pi\frac{(r_++r_-)^{3/2}}{\sqrt{2Gm}}\left\{1+\frac{3(1+\lambda)\,Gm}{r_++r_-}+6\frac{Gm}{r_++r_-}\left[1-\left(\frac{r_-}{r_+}\right)^{1/2}\frac{r_++r_-}{2r_+}\right]\right\}. \tag{18.117}$$

We see that the period of revolution of test bodies within the GR framework is not determined unambiguously because the value of λ is not fixed in GR. On the other hand, in RTG $\lambda = 0$, so that the predictions of RTG concerning the period of revolution of a test body are unambiguous:

$$T\simeq\pi\frac{(r_++r_-)^{3/2}}{\sqrt{2Gm}}\left\{1+\frac{9Gm}{r_++r_-}\left[1-\left(\frac{r_-}{r_+}\right)^{1/2}\frac{r_++r_-}{3r_+}\right]\right\}.$$

The last formula reflects Kepler's third law in RTG with corrections.

18.7 Shirokov's Effect*

Shirokov's effect is that if a test particle moving in a spherically symmetric gravitational field along a closed orbit is acted upon by a weak perturbation, it will oscillate both radially and azimuthally. This problem is of interest to us from the

* See Shirokov, 1973.

methodological angle. Hence, we will consider a simplified version: we assume that the test particle moves uniformly along a circular orbit of radius $r = \text{const}$.

To determine the infinitesimal 4-vector ξ^i of a deviation from the geodesic, we start with the deviation equation

$$\frac{D^2\xi^i}{Ds^2} + R^i_{jkl}u^j u^l \xi^k = 0, \tag{18.118}$$

where s is the parameter of the trajectory, and

$$u^i = \frac{dx^i}{ds}, \tag{18.119}$$

with x^i the spherical coordinates, that is, $x^0 = t$, $x^1 = r$, $x^2 = \theta$, and $x^3 = \varphi$. Obviously, u^i is the 4-vector of the rate of change of the unperturbed coordinates x^i with respect to parameter s and satisfies the following equation of motion (the geodesic equation):

$$\frac{du^i}{ds} + \Gamma^i_{jk}u^j u^k = 0. \tag{18.120}$$

Moving the origin of the coordinate system to the center of the source of gravitational field and superposing the equatorial plane xy with the plane in which the test particle moves, we get $\theta = \pi/2$, which means that

$$\frac{d\theta}{ds} \equiv u^2 = 0. \tag{18.121}$$

Since, by assumption, the orbit is circular, $r = \text{const}$, and we have

$$u^1 \equiv \frac{dr}{ds} = 0. \tag{18.122}$$

Allowing for the uniformity of the motion and the validity of (18.29), (18.121), and (18.122), we find that (18.120) yields

$$\frac{dU}{dr}(u^0)^2 - \frac{dW}{dr}(u^3)^2 = 0. \tag{18.123}$$

Since

$$g_{mn}u^m u^n = 1,$$

and (18.25), (18.121), and (18.122) are valid, we obtain

$$U(r)(u^0)^2 - W(r)(u^3)^2 = 1, \tag{18.124}$$

which, when combined with (18.123), can be used to find $(u^0)^2$ and $(u^3)^2$:

$$(u^0)^2 = \frac{1}{UW}\frac{dW}{dr}\frac{1}{\frac{d}{dr}\ln\left(\frac{W}{U}\right)}, \tag{18.125}$$

$$(u^3)^2 = \frac{1}{UW}\frac{dU}{dr}\frac{1}{\frac{d}{dr}\ln\left(\frac{W}{U}\right)}. \tag{18.126}$$

Let us now return to the deviation equation (18.118). An identical form of this equation is

$$\frac{d^2\xi^i}{ds^2} + 2\Gamma^i_{mn}u^m\frac{d\xi^n}{ds} + \frac{\partial\Gamma^i_{mn}}{\partial x^p}\xi^p u^m u^n = 0. \tag{18.127}$$

This combined with (18.29) and with the fact that $\theta = \pi/2$ yields

$$\frac{d^2\xi^0}{ds^2} + a\frac{d\xi^1}{ds} = 0, \tag{18.128}$$

$$\frac{d^2\xi^1}{ds^2} + b\frac{d\xi^0}{ds} + c\frac{d\xi^3}{ds} + f\xi^1 = 0, \tag{18.129}$$

$$\frac{d^2\xi^2}{ds^2} + e\xi^2 = 0, \tag{18.130}$$

$$\frac{d^2\xi^3}{ds^2} + k\frac{d\xi^1}{ds} = 0, \tag{18.131}$$

where we have introduced the notation

$$\begin{gathered} a = 2u^0\Gamma^0_{01}, \quad b = 2\Gamma^1_{00}u^0, \quad c = 2\Gamma^1_{22}u^3, \\ f = \frac{d\Gamma^1_{00}}{dr}(u^0)^2 + \frac{d\Gamma^1_{22}}{dr}(u^3)^2, \quad e = (u^3)^2, \quad k = 2\Gamma^3_{13}u^3. \end{gathered} \tag{18.132}$$

Note that in the current problem the coefficients a, b, c, f, e, and k are constants.

Equation (18.130) yields

$$\xi^2 = \xi^2_0 \exp(i\Omega s), \tag{18.133}$$

where $\Omega^2 = e$ and, by virtue of (18.132) and (18.126),

$$\Omega^2 = \frac{1}{W}\frac{d\ln U}{dr}\frac{1}{\frac{d}{dr}\ln\left(\frac{W}{U}\right)}. \tag{18.134}$$

If in the remaining equations, (18.128), (18.129), and (18.131), we carry out the substitutions

$$\xi^0 = \xi^0_0 \exp(i\omega s), \quad \xi^1 = \xi^1_0 \exp(i\omega s), \quad \xi^3 = \xi^3_0 \exp(i\omega s), \tag{18.135}$$

we obtain

$$\begin{gathered} -\omega^2\xi^0_0 + ia\omega\xi^1_0 = 0, \\ -\omega^2\xi^1_0 + ib\omega\xi^0_0 + ic\omega\xi^3_0 + f\xi^1_0 = 0, \\ -\omega^2\xi^3_0 + i\omega k\xi^1_0 = 0. \end{gathered} \tag{18.136}$$

The condition that there be nontrivial solutions to these equations yields

$$\omega^2 = f - kc - ab. \tag{18.137}$$

Suppose that we have taken the one-parameter family

$$W(r) = [r + (1+\lambda)Gm]^2$$

as the solution to the Hilbert-Einstein equations, with λ an adjustable parameter and m the active gravitational mass of the source of gravitational field. Then combining (18.134) with the relationship (12.40) that links U with $\sqrt{W}$, we obtain

$$\Omega^2 = \frac{Gm}{[r+(\lambda-2)Gm][r+(\lambda+1)Gm]^2}. \tag{18.138}$$

On the basis of (12.40), (18.29), (18.132), and (18.137) we can write the following formula for ω^2:

$$\omega^2 = \Omega^2\left[2\,\frac{r+(\lambda-2)Gm}{r+(1+\lambda)Gm} - 1\right]. \tag{18.139}$$

For all values of λ satisfying the conditions

$$r \gg Gm\,|\lambda+1|, \quad r \gg Gm\,|\lambda-2|, \tag{18.140}$$

formulas (18.138) and (18.139) yield

$$\Omega^2 \simeq \Omega_0^2 \left(1 - \frac{3\lambda Gm}{r}\right), \tag{18.141}$$

$$\omega^2 \simeq \Omega_0^2 \left(1 - \frac{3(\lambda+2)\, Gm}{r}\right), \tag{18.142}$$

with

$$\Omega_0^2 = Gm/r^3. \tag{18.143}$$

This results in the following formulas for the periods of radial and azimuthal oscillations in the first order in Gm:

$$T_r = \frac{2\pi}{\omega} \simeq \frac{2\pi}{\Omega_0}\left[1 + \frac{3(\lambda+2)\, Gm}{2r}\right], \tag{18.144}$$

$$T_\theta = T_\varphi = \frac{2\pi}{\Omega} = \frac{2\pi}{\Omega_0}\left[1 + \frac{3\lambda Gm}{2r}\right]. \tag{18.145}$$

We see that in GR in the very first order in Gm, the formulas do not provide an unambiguous answer but that in RTG, where $\lambda = 0$, we have well-defined values for T_r and $T_\theta = T_\varphi$:

$$T_r = \frac{2\pi}{\Omega_0}\left(1 + \frac{3Gm}{r}\right), \tag{18.146}$$

$$T_\theta = T_\varphi = \frac{2\pi}{\Omega_0}. \tag{18.147}$$

18.8 Precession of a Gyroscope in Orbit

Suppose that a gyroscope is moving along a closed orbit around a massive object. Then the change in the 4-vector of the intrinsic angular momentum S_n of the gyroscope is described by the equation

$$\frac{dS_n}{ds} = \Gamma^k_{mn} S_k \frac{dx^m}{ds}, \tag{18.148}$$

where $x^m = (t, x^\alpha)$, with x^α being the spherical coordinates.

Since this problem interests us only from the standpoint of the ambiguity of RTG and GR predictions, to simplify matters we will assume that the gyroscope follows a circular orbit in a spherically symmetric gravitational field and that it is point-like. Hence, it is advisable to use the term "spin" in relation to the 4-vector of the intrinsic angular momentum S_n of the gyroscope. Spin S_n is orthogonal to the 4-vector $u^m = dx^m/ds$ of the velocity of the gyroscope in orbit, and, therefore

$$u^0 S_0 = -u^\alpha S_\alpha. \tag{18.149}$$

Setting $n = \alpha$ in (18.148) and allowing for (18.149), we get

$$\frac{dS_\alpha}{ds} = -\Gamma^0_{0\alpha} S_\beta u^\beta - \Gamma^0_{\nu\alpha} S_\beta \frac{u^\beta u^\nu}{u^0} + \Gamma^\beta_{0\alpha} S_\beta u^0 + \Gamma^\beta_{\nu\alpha} S_\beta u^\nu. \tag{18.150}$$

Let us place the origin of the coordinate system at the center of the source of gravitational field and superpose the equatorial plane with the plane in which the gyroscope moves. In this case, obviously, $\theta = \pi/2$ and therefore

$$u^2 = \frac{d\theta}{ds} = 0.$$

Since by assumption the orbit is a circle of radius $r = \text{const}$, we have

$$u^1 = \frac{dr}{ds} = 0,$$

with the result that the 4-vector of velocity, u^m, assumes the form

$$u^m = (u^0, 0, 0, u^3). \tag{18.151}$$

If we now take into account the expressions (18.29) for the connection coefficients Γ^k_{mn}, the fact that $\theta = \pi/2$, and (18.151), we will find that (18.150) yields

$$\frac{dS_1}{ds} = -(\Gamma^0_{01} - \Gamma^3_{31})(S_\alpha u^\alpha), \tag{18.152}$$

$$\frac{dS_2}{ds} = 0, \tag{18.153}$$

$$\frac{dS_3}{ds} = \Gamma^1_{33} S_1 u^3. \tag{18.154}$$

We see that the component S_2 is conserved.

For the trajectory considered here we already know the velocity components u^0 and u^3. The values are given by (18.125) and (18.126), respectively.

Multiplying (18.152)-(18.154) by $(u^0)^{-1}$ and introducing the notation

$$v^\alpha = u^\alpha/u^0 = dx^\alpha/dt, \tag{18.155}$$

we obtain

$$\frac{dS_1}{dt} = -(\Gamma^0_{01} - \Gamma^3_{31}) S_3 v^3, \tag{18.156}$$

$$\frac{dS_2}{dt} = 0, \tag{18.157}$$

$$\frac{dS_3}{dt} = \Gamma^1_{33} S_1 v^3. \tag{18.158}$$

Obviously, $v^1 = v^2 = 0$, while for v^3, in view of (18.125) and (18.126), we have

$$v^3 = \left(\frac{1}{2\sqrt{W}} \frac{dU}{d\sqrt{W}}\right)^{1/2}. \tag{18.159}$$

Now let us prove that Eqs. (18.156)-(18.158) imply that the scalar product of the 4-vectors S_m,

$$S_m S^m = g^{mn} S_m S_n, \tag{18.160}$$

is time independent and, therefore, is conserved. Allowing for (18.25), (18.149), (18.155), and (18.159), we find that (18.160) with $\theta = \pi/2$ yields

$$S_m S^m = -\frac{1}{V}(S_1)^2 - \frac{1}{W}(S_2)^2 - \left(\frac{1}{W} - \frac{1}{2U\sqrt{W}} \frac{dU}{d\sqrt{W}}\right)(S_3)^2. \tag{18.161}$$

Finding the time derivative of this expression and allowing for Eqs. (18.156)-(18.158) yields

$$\frac{d}{dt}(S_m S^m) = 2v^3 S_1 S_3 \left[\frac{1}{V}(\Gamma^0_{01} - \Gamma^3_{13}) - \Gamma^1_{33}\left(\frac{1}{W} - \frac{1}{2U\sqrt{W}} \frac{dU}{d\sqrt{W}}\right)\right].$$

If we employ (18.29), we can demonstrate that

$$\frac{1}{V}(\Gamma^0_{01} - \Gamma^3_{13}) \equiv \Gamma^1_{33}\left(\frac{1}{W} - \frac{1}{2U\sqrt{W}} \frac{dU}{d\sqrt{W}}\right), \tag{18.162}$$

whereby

$$\frac{d}{dt}(S_m S^m) \equiv 0, \tag{18.163}$$

which is what we set out to prove.

Let us introduce the 3-vector $\mathbf{Z}$ with components

$$Z_1 = \frac{1}{\sqrt{V}} S_1, \quad Z_2 = -\frac{r}{\sqrt{W}} S_2,$$
$$Z_3 = -r \sin\theta \left(\frac{1}{W} - \frac{1}{2U\sqrt{W}} \frac{dU}{d\sqrt{W}} \right)^{1/2} S_3 \tag{18.164}$$

with respect to spherical coordinates. It can easily be shown that

$$-(\mathbf{Z}\cdot\mathbf{Z}) = \gamma^{\alpha\beta} Z_\alpha Z_\beta = (S_m S^m), \tag{18.165}$$

which implies, with the aid of (18.163), that $|\mathbf{Z}|$ is conserved.

We now wish to find the Cartesian components of $\mathbf{Z}$. To this end we employ the vector transformation law

$$Z'_\beta = \frac{\partial x^\alpha}{\partial x'^\beta} Z_\alpha, \tag{18.166}$$

where the primes on the symbols indicate that the respective quantities refer to a Cartesian system of coordinates.

Since the motion occurs in the (x'^1, x'^2) plane (i.e. $\theta = \pi/2$), the transformation matrix $\partial x^\alpha / \partial x'^\beta$ has the form

$$\frac{\partial x^\alpha}{\partial x'^\beta} = \begin{pmatrix} \frac{x'^1}{r} & 0 & -\frac{x'^2}{r^2} \\ \frac{x'^2}{r} & 0 & \frac{x'^1}{r^2} \\ 0 & -\frac{1}{r} & 0 \end{pmatrix}$$

with the result that (18.164) yields

$$Z'_1 = \frac{1}{r} \left(x'^1 Z_1 - \frac{x'^2}{r} Z_3 \right), \tag{18.167}$$

$$Z'_2 = \frac{1}{r} \left(x'^2 Z_1 + \frac{x'^1}{r} Z_3 \right), \tag{18.168}$$

$$Z'_3 = -\frac{1}{r} Z_2. \tag{18.169}$$

Substituting the expressions (18.164) into (18.167)-(18.169) with $\theta = \pi/2$, we obtain

$$Z'_1 = \frac{1}{r} \left[\frac{x'^1}{\sqrt{V}} S_1 + x'^2 \left(\frac{1}{W} - \frac{1}{2U\sqrt{W}} \frac{dU}{d\sqrt{W}} \right)^{1/2} S_3 \right], \tag{18.170}$$

$$Z'_2 = \frac{1}{r} \left(\frac{x'^2}{\sqrt{V}} S_1 - x'^1 \left(\frac{1}{W} - \frac{1}{2U\sqrt{W}} \frac{dU}{d\sqrt{W}} \right)^{1/2} S_3 \right], \tag{18.171}$$

$$Z'_3 = \frac{1}{\sqrt{W}} S_2. \tag{18.172}$$

Let us denote the Cartesian components of the velocity vector $\mathbf{v}'$ by v'^1 and v'^2:

$$v'^1 = \frac{dx'^1}{dt}, \quad v'^2 = \frac{dx'^2}{dt}.$$

It is easily established that v'^1 and v'^2 are related to $v^3 = d\varphi/dt$ in the following manner:

$$v'^1 = -x'^2 v^3, \quad v'^2 = x'^1 v^3. \tag{18.173}$$

Differentiating (18.170)-(18.172) with respect to t and allowing for Eqs. (18.156)-(18.158) and for (18.173), we find that

$$\frac{dZ_1'}{dt}=\frac{1}{r}\left\{S_1 v'^1\left[\frac{1}{\sqrt{V}}-\Gamma^1_{33}\left(\frac{1}{W}-\frac{1}{2U\sqrt{W}}\frac{dU}{d\sqrt{W}}\right)^{1/2}\right]\right.$$
$$\left.+S_3 v'^2\left[\left(\frac{1}{W}-\frac{1}{2U\sqrt{W}}\frac{dU}{d\sqrt{W}}\right)^{1/2}-\frac{1}{\sqrt{V}}(\Gamma^0_{01}-\Gamma^3_{31})\right]\right\}, \quad (18.174)$$

$$\frac{dZ_2'}{dt}=\frac{1}{r}\left\{v'^2 S_1\left[\frac{1}{\sqrt{V}}-\Gamma^1_{33}\left(\frac{1}{W}-\frac{1}{2U\sqrt{W}}\frac{dU}{d\sqrt{W}}\right)^{1/2}\right]\right.$$
$$\left.+v'^1 S_3\left[\left(\frac{1}{W}-\frac{1}{2U\sqrt{W}}\frac{dU}{d\sqrt{W}}\right)^{1/2}-\frac{1}{\sqrt{V}}(\Gamma^0_{01}-\Gamma^3_{31})\right]\right\}, \quad (18.175)$$

$$\frac{dZ_3'}{dt}=0. \quad (18.176)$$

Solving Eqs. (18.170)-(18.172) for S_1, S_2, and S_3 and substituting these into (18.174)-(18.176) and allowing for the identity (18.162) yields

$$\frac{dZ_1'}{dt}=\frac{Z_2'}{r^2}\left[1-(\Gamma^1_{33}\Gamma^0_{01}-\Gamma^1_{33}\Gamma^3_{13})^{1/2}\right](v'^1x'^2-v'^2x'^1), \quad (18.177)$$

$$\frac{dZ_2'}{dt}=\frac{Z_1'}{r^2}\left[1-(\Gamma^1_{33}\Gamma^0_{01}-\Gamma^1_{33}\Gamma^3_{13})^{1/2}\right](v'^2x'^1-v'^1x'^2), \quad (18.178)$$

$$\frac{dZ_3'}{dt}=0. \quad (18.179)$$

Here, in deriving (18.177) and (18.178), we have allowed for the fact that the velocity vector $\mathbf{v}'$ is orthogonal to the radius vector $\mathbf{x}'$.

It can be seen that (18.177)-(18.179) can be written in vector form:

$$\frac{d\mathbf{Z}'}{dt}=\frac{1}{r}(\mathbf{Z}'\times\mathbf{\Omega}), \quad (18.180)$$

where

$$\mathbf{\Omega}=\left[1-(\Gamma^1_{33}\Gamma^0_{01}-\Gamma^1_{33}\Gamma^3_{13})^{1/2}\right]\left(\frac{\mathbf{x}'}{r}\times\mathbf{v}'\right). \quad (18.181)$$

Equation (18.180) shows that vector $\mathbf{Z}'$ precesses around vector $\mathbf{\Omega}$ with a rate $|\mathbf{\Omega}|$.

If for the solution to the Hilbert-Einstein equation we take the one-parameter family of functions

$$W(r)=[r+(1+\lambda)Gm]^2, \quad (18.182)$$

where λ is an adjustable parameter, and m the active gravitational mass of the source, then we find that

$$|\mathbf{\Omega}|=|\mathbf{v}'|\,|1-(\Gamma^1_{33}\Gamma^0_{01}-\Gamma^1_{33}\Gamma^3_{13})^{1/2}|=|\mathbf{v}'|\left|1-\left(\frac{r+(\lambda-2)Gm}{r+(\lambda+1)Gm}\right)^{1/2}\right|. \quad (18.183)$$

If the radius of the gyroscope's orbit is much greater than Gm, for all values of λ satisfying the condition $r\gg|\lambda+1|\,Gm$ Eq. (18.183) yields, to the second order in Gm/r, the following expansion:

$$|\mathbf{\Omega}|\simeq\frac{3}{2}\frac{Gm}{r}|\mathbf{v}'|\left|1-\frac{Gm}{r}\left(\lambda+\frac{1}{4}\right)\right|. \quad (18.184)$$

This expansion shows that the ambiguity, due to the GR approach, in the expression for $|\mathbf{\Omega}|$ manifests itself in the second order in Gm/r, while the value of $|\mathbf{\Omega}|$ in RTG is well-defined because here $\lambda=0$.

18.9 GR and Gravitational Effects in the Solar System. Conclusion

Generally, the properties of space-time can be established in experiments. Indeed, suppose that we know the equations of all the timelike geodesics and all the isotropic geodesics in a selected system of coordinates. Then the space-time metric tensor in this system is determined to within a constant factor (for the proof of this statement see Petrov, 1966). But from the physical viewpoint the geodesics are the lines along which test particles move. Hence, by studying light propagation and the motion of test bodies it is possible in principle to determine experimentally the geometry of the effective Riemann space-time.

Can we determine the geometry of space-time if we remain in the GR framework? The Hilbert-Einstein equation in a chosen system of coordinates is valid, generally speaking, for any arbitrary functions of four metric coefficients. In other words, Hilbert-Einstein equations do not determine the Riemannian geometry. The common approach in GR to determining the four unknown metric coefficients in a given coordinate system is to use the so-called coordinate conditions, which are four additional noncovariant equations for the metric coefficients.

The Hilbert-Einstein equations and the coordinate conditions enable us to determine, in a given system of coordinates, the Riemannian geometry of space-time and to calculate this or another gravitational effect. The choice of the coordinate conditions in GR is completely arbitrary, and it is assumed that this choice does not affect the physical results in a given system of coordinates. But this choice does influence the functional form of the metric coefficients and, hence, depending on the form of the coordinate conditions in a given system of coordinates, we have different geometries of the Riemann space-time. And in view of the above statement different Riemannian geometries lead to different predictions concerning the propagation of light and the motion of test bodies. All of this has been demonstrated using solar-system gravitational effects as examples.

Summing up, we can say that GR is incapable, no matter what the advocates of this ideology may say, of making definite predictions concerning the geometry of the Riemann space-time and gravitational effects. This constitutes still another important defect of GR.

Chapter 19. Post-Newtonian Integrals of Motion in RTG

The covariant conservation law (6.28) for the total energy-momentum tensor density t^{mn} in the pseudo-Euclidean space-time can be written in terms of Cartesian coordinates thus:

$$\partial_m t^{mn} = \partial_m \left[t^{mn}_{(g)} + t^{mn}_{(M)}\right] = 0. \tag{19.1}$$

On the basis of this law we can easily derive the respective integral conservation law

$$-\partial_0 \int t^{0n}\, dV = \oint t^{\alpha n}\, dS_\alpha. \tag{19.2}$$

If the energy flux for matter and gravitational field across the surface bounding the volume of integration is nil in (19.2), then

$$\oint t^{\alpha n}\, dS_\alpha = 0 \tag{19.3}$$

and we arrive at the law of conservation of the total 4-momentum of an isolated system:

$$\frac{dP^n}{dt}=0,$$

where

$$P^n=\int t^{0n}\,dV. \tag{19.4}$$

In this case, in view of the symmetry of the total energy-momentum tensor density t^{mn}, the angular momentum tensor of the system is also conserved:

$$\frac{d}{dt}M^{ni}=0,$$

where

$$M^{ni}=\int[x^n t^{0i}-x^i t^{0n}]\,dV. \tag{19.5}$$

The fact that the $M^{0\alpha}$ components are conserved implies that the center of mass of an isolated system, specified by the formula

$$X^\alpha=\frac{\int x^\alpha t^{00}\,dV}{\int t^{00}\,dV}=\frac{P^\alpha t-M^{0\alpha}}{P^0}, \tag{19.6}$$

is in uniform rectilinear motion with a velocity

$$\frac{d}{dt}X^\alpha=\frac{P^\alpha}{P^0}. \tag{19.6'}$$

Thus, to describe the motion of an isolated system consisting of matter and gravitational field it is sufficient to determine the 4-momentum P^n (19.4). Note that a real system cannot be strictly isolated because of the motion of its constituent parts, which causes emission of gravitational waves, and because a real system exchanges matter with other systems in the form of electromagnetic radiation, particles, atoms, and the like. Hence, in general, we cannot ignore the energy fluxes for matter and gravitational field. There are astrophysical processes in which these energy fluxes play the leading role. Taking them into account enables us to understand and predict many an astrophysical phenomenon. At the same time, however, for systems within which the energy fluxes for matter and gravitational field are small, condition (19.3) is met with a certain degree of accuracy. Then, with the same degree of accuracy we can state that the 4-momentum of such a system is conserved. These are the systems to which the post-Newtonian formalism can be applied.

To find the explicit form of the 4-momentum P^n in the post-Newtonian approximation, let us turn to the identity (6.26). We write this identity in a Cartesian system of coordinates:

$$\partial_m t_n^m=\nabla_m T_n^m. \tag{19.7}$$

Multiplying both sides of (19.7) by dV and integrating over a sufficiently large volume, we obtain

$$\partial_0\int t_n^0\,dV+\oint t_n^\alpha\,dS_\alpha=\int\nabla_m T_n^m\,dV. \tag{19.8}$$

Assuming that

$$\oint t_n^\alpha\,dS_\alpha=0, \tag{19.9}$$

we find that

$$\partial_0 P_n = \int \nabla_m T_n^m \, dV, \tag{19.10}$$

where in view of the definition (19.4)

$$P_n = \gamma_{nk} \int t^{0k} \, dV. \tag{19.11}$$

Equation (19.10) is exact to the same degree as the energy flux of matter and gravitational field across the surface bounding the integration volume in (19.18) is close to zero, that is, as (19.9) is valid.

Let us now employ Eq. (19.10) to determine the explicit form of the 4-momentum P_n. This can be done if we represent the right-hand side of (19.10) in the form of the time derivative of a certain expression. Since our goal in this chapter is to find the integrals of motion in the post-Newtonian approximation, it is natural to consider the right-hand side of (19.10) in this approximation. Let us write the expressions for each component of $\nabla_m T^{mk}$ for the energy-momentum tensor density of matter:

$$\nabla_m T^{m0} = \partial_0 T^{00} + \partial_\alpha T^{\alpha 0} + \Gamma_{00}^0 T^{00} + 2\Gamma_{0\alpha}^0 T^{\alpha 0} + \Gamma_{\alpha\beta}^0 T^{\alpha\beta}, \tag{19.12}$$

$$\nabla_m T^{m\alpha} = \partial_0 T^{0\alpha} + \partial_\nu T^{\nu\alpha} + \Gamma_{00}^\alpha T^{00} + 2\Gamma_{0\nu}^\alpha T^{\nu 0} + \Gamma_{\beta\nu}^\alpha T^{\beta\nu}, \tag{19.13}$$

where

$$\Gamma_{pq}^k = \frac{1}{2} g^{kl} (\partial_p g_{lq} + \partial_q g_{lp} - \partial_l g_{pq}). \tag{19.14}$$

Using the post-Newtonian expansion of the metric, (17.72)-(17.74), and the fact that

$$\partial^\alpha \partial_0 \int \rho \, | \mathbf{x} - \mathbf{x}' | \, dV = N^\alpha - V^\alpha + O(\varepsilon^5),$$

we find that (19.14) yields, to required accuracy,

$$\begin{gathered}
\Gamma_{00}^0 = -\frac{\partial U}{\partial t} + O(\varepsilon^5), \quad \Gamma_{0\alpha}^0 = -\partial_\alpha U + O(\varepsilon^4), \quad \Gamma_{\alpha\beta}^0 = O(\varepsilon^3), \\
\Gamma_{\beta\nu}^\alpha = (\delta_\beta^\alpha \partial_\nu U + \delta_\nu^\alpha \partial_\beta U - \gamma^{\alpha\sigma}\gamma_{\beta\nu}\partial_\sigma U) + O(\varepsilon^4), \\
\Gamma_{0\beta}^\alpha = \delta_\beta^\alpha \frac{\partial U}{\partial t} + 2(\partial_\beta V^\alpha - \gamma^{\alpha\sigma}\gamma_{\beta\tau}\partial_\sigma V^\tau) + O(\varepsilon^5), \\
\Gamma_{00}^\alpha = 4\,\frac{\partial V^\alpha}{\partial t} + (1 - 2U)\,\partial^\alpha U - \frac{1}{2}\partial^\alpha [2U^2 - 4\Phi_1 - 4\Phi_2 - 2\Phi_3 - 6\Phi_4] \\
+ \frac{1}{2}\frac{\partial}{\partial t}(N^\alpha - V^\alpha) + O(\varepsilon^6).
\end{gathered} \tag{19.15}$$

Moreover, with post-Newtonian accuracy Eqs. (17.51)-(17.53), (17.67) and (17.68) yield

$$\begin{gathered}
T^{00} = \hat{\rho}\left(1 + U + \Pi - \frac{1}{2} v_\nu v^\nu\right) + O(\varepsilon^4), \\
T^{0\alpha} = \hat{\rho} v^\alpha \left[1 + \Pi + U - \frac{1}{2} v_\nu v^\nu\right] + p v^\alpha + O(\varepsilon^5), \\
T^{\alpha\beta} = \hat{\rho} v^\alpha v^\beta \left(1 - \frac{1}{2} v_\nu v^\nu + \Pi + U + p/\hat{\rho}\right) - \gamma^{\alpha\beta} p + O(\varepsilon^6).
\end{gathered} \tag{19.16}$$

Substituting (19.15) and (19.16) into (19.12), we obtain

$$\nabla_m T^{m0} = \frac{\partial}{\partial t}\left[\hat{\rho}\left(1+U+\Pi-\frac{1}{2}v_\nu v^\nu\right)\right] + \partial_\alpha\left[\hat{\rho}v^\alpha\left(1+U+\Pi-\frac{1}{2}v_\nu v^\nu\right)+pv^\alpha\right] - \hat{\rho}\frac{\partial U}{\partial t} - 2\hat{\rho}v^\alpha\partial_\alpha U + O(\varepsilon^5). \quad (19.17)$$

Similarly, substitution of (19.15) and (19.16) into (19.13) yields

$$\begin{aligned}\nabla_m T^{m\alpha} = {} & \frac{\partial}{\partial t}\left[\hat{\rho}v^\alpha\left(1+\Pi+U-\frac{1}{2}v_\nu v^\nu\right)+pv^\alpha\right] \\ & + \partial_\beta\left[pv^\alpha v^\beta - p\gamma^{\alpha\beta} + \hat{\rho}v^\alpha v^\beta\left(1+\Pi+U-\frac{1}{2}v_\nu v^\nu\right)\right] \\ & + \frac{7}{2}\hat{\rho}\frac{\partial V^\alpha}{\partial t} + \hat{\rho}\left(1+\Pi+U-\frac{1}{2}v_\nu v^\nu\right)\partial^\alpha U \\ & - 4\hat{\rho}U\partial^\alpha U + \hat{\rho}\partial^\alpha(2\Phi_1+2\Phi_2+\Phi_3+3\Phi_4) + 2\hat{\rho}v^\alpha\frac{\partial U}{\partial t} \\ & + \hat{\rho}\frac{1}{2}\frac{\partial N^\alpha}{\partial t} + 4\hat{\rho}v^\beta(\partial_\beta V^\alpha - \partial^\alpha V_\beta) + p\partial^\alpha U \\ & + 2\hat{\rho}v^\alpha v^\beta\partial_\beta U + \hat{\rho}v^2\partial^\alpha U + O(\varepsilon^6). \end{aligned} \quad (19.18)$$

The formulas just obtained, (19.17) and (19.18), can be simplified if we take into account the continuity equation (17.49) and the Newtonian laws of motion of an elastic body,

$$\hat{\rho}\frac{dv^\alpha}{dt} = -\hat{\rho}\partial^\alpha U + \partial^\alpha p, \qquad \rho\frac{d\Pi}{dt} = -p\partial_\nu v^\nu, \quad (19.19)$$

with

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v_\beta\partial^\beta,$$

and the following relationships:

$$\partial_\beta V_\alpha - \partial_\alpha V_\beta = \partial_\beta N_\alpha - \partial_\alpha N_\beta, \qquad \hat{\rho}\frac{\partial U}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(\hat{\rho}U) + \frac{1}{8\pi}\partial^\alpha\left[\partial_\alpha U\frac{\partial U}{\partial t} - U\partial_\alpha\frac{\partial U}{\partial t}\right]. \quad (19.20)$$

The result is

$$\begin{aligned}\nabla_m T^{m0} = {} & \frac{\partial}{\partial t}\left[\hat{\rho}\left(1+\Pi-\frac{1}{2}U-\frac{1}{2}v_\nu v^\nu\right)\right] \\ & + \partial_\alpha\left[\hat{\rho}v^\alpha\left(1+\Pi+U-\frac{1}{2}v_\nu v^\nu\right)+pv^\alpha \right. \\ & \left. - \frac{1}{8\pi}\partial^\alpha U\frac{\partial U}{\partial t} + \frac{1}{8\pi}U\partial^\alpha\frac{\partial U}{\partial t}\right] + O(\varepsilon^4), \end{aligned} \quad (19.21)$$

$$\begin{aligned}\nabla_m T^{m\alpha} = {} & \frac{\partial}{\partial t}\left[\hat{\rho}v^\alpha\left(1+\Pi+U-\frac{1}{2}v_\nu v^\nu\right)+pv^\alpha+\frac{1}{2}\hat{\rho}(N^\alpha-V^\alpha)\right] \\ & + \partial_\beta\left[\hat{\rho}v^\alpha v^\beta\left(1+\Pi+U-\frac{1}{2}v_\nu v^\nu\right)+pv^\alpha v^\beta\right.\end{aligned}$$

$$
\begin{aligned}
&- p\,(1+2U)\,\gamma^{\alpha\beta} + \hat{\rho} v^{\beta}\,(N^{\alpha} - V^{\alpha})\Big] \\
&+ 4\hat{\rho}\,\frac{dV^{\alpha}}{dt} + 2\,\hat{\rho}\,\frac{d}{dt}\,(Uv^{\alpha}) + \hat{\rho}\partial^{\alpha}U + 2\hat{\rho}U\partial^{\alpha}U \\
&+ \hat{\rho}\,(\Pi + 2v^2 + 3p/\hat{\rho})\,\partial^{\alpha}U - 4\hat{\rho}v^{\beta}\partial^{\alpha}V_{\beta} - \frac{1}{2}\,\hat{\rho}v^{\beta}\partial_{\beta}\,(N^{\alpha} - V^{\alpha}) \\
&+ \hat{\rho}\partial^{\alpha}\,[2\Phi_1 + 2\Phi_2 + \Phi_3 + 3\Phi_4] + O\,(\varepsilon^6).
\end{aligned}
\tag{19.22}
$$

Substituting this into the right-hand side of (19.10), allowing for the identities

$$
\begin{gathered}
\int \rho\,[\Pi\partial_{\beta}U + \partial_{\beta}\Phi_3]\,dV = 0, \\
\int \rho\,[U\partial_{\beta}U + \partial_{\beta}\Phi_2]\,dV = 0, \\
\int [p\partial_{\beta}U + \rho\partial_{\beta}\Phi_4]\,dV = 0, \\
\int \rho\,[V^2\partial_{\beta}U + \partial_{\beta}\Phi_1]\,dV = 0, \\
\int \rho v^{\beta}\partial_{\beta}\,(N^{\alpha} - V^{\alpha})\,dV = 0, \\
\int \frac{\partial U}{\partial t}\,\partial^{\alpha}U\,dV = 2\int \rho\,(v^{\alpha}U + N^{\alpha})\,dV = 0, \\
\int \rho V^{\alpha}\,dV = -\int \rho v^{\alpha}U\,dV, \\
\int \rho\partial_{\beta}U\,dV = 0, \\
\int \rho v_{\nu}\partial_{\beta}V^{\nu}\,dV = \int \rho v_{\nu}\partial_{\beta}N^{\nu}\,dV = 0,
\end{gathered}
\tag{19.23}
$$

and taking into account the fact that the volume integrals of spatial divergence vanish after transformation into surface integrals, we obtain

$$
\frac{\partial}{\partial t}\,P^0 = \frac{\partial}{\partial t}\int \hat{\rho}\,\left(1 + \Pi - \frac{1}{2}\,U - \frac{1}{2}\,v_{\nu}v^{\nu}\right)\,dV, \tag{19.24}
$$

$$
\frac{\partial}{\partial t}\,P_{\alpha} = \frac{\partial}{\partial t}\int \left[\hat{\rho}v_{\alpha}\,\left(1 + \Pi - \frac{1}{2}\,U - \frac{1}{2}\,v_{\nu}v^{\nu}\right) + pv_{\alpha} + \frac{1}{2}\,\hat{\rho}N_{\alpha}\right]dV. \tag{19.25}
$$

In deriving the last formula we employed the fact that the lowering and raising of indices in (19.22) can be achieved by using the metric tensor of the Minkowski space-time. The final expressions emerging from (19.24) and (19.25) are

$$
P^0 = \int \hat{\rho}\,\left[1 + \Pi - \frac{1}{2}\,U - \frac{1}{2}\,v_{\nu}v^{\nu}\right]dV, \tag{19.26}
$$

$$
P^{\alpha} = \int \left[\hat{\rho}v^{\alpha}\left(1 + \Pi - \frac{1}{2}\,U - \frac{1}{2}\,v_{\nu}v^{\nu}\right) + pv^{\alpha} + \frac{1}{2}\,\hat{\rho}N^{\alpha}\right]dV. \tag{19.27}
$$

Chapter 20. Do Extended Objects Move Along Geodesics in the Riemann Space-Time?

In Chapter 17 we gave formulas (17.75)-(17.77) for the metric coefficients of the Riemann space-time in terms of sums of various generalized potentials with ten arbitrary coefficients β, γ, α_1, α_2, α_3, ξ_1, ξ_2, ξ_3, ξ_4, and ξ_w, known as post-Newtonian parameters (the parametrized post-Newtonian, or PPN, formalism). To each theory of gravitation in this formalism there corresponds a definite set of post-Newtonian parameters. Hence, by determining the values of these parameters we can select from all the gravitational experiments only those theories whose post-Newtonian approximation leads to values of the parameters coinciding with those obtained from experiments.

In this chapter we first consider the ratio of the gravitational mass to the inertial mass and establish the relationships between PPN parameters that follow from the requirement that there must be post-Newtonian laws of conservation. Then we study the equations of motion of the center of mass of an extended object, the equations of geodesic motion of a point-like object in the vicinity of the Earth's center of mass, the deviation of the Earth's center-of-mass motion from the reference geodesic, and a number of questions pertaining to the problem of the motion of objects along geodesics. We also analyze the motion of the Earth's center of mass in gravimetric experiments and list the effects connected with the presence of a preferred reference frame and those associated with the anisotropy in relation to the center of the Galaxy. By investigating these problems we can arrive at bounds on the values of PPN parameters.

The problem of the ratio of the gravitational mass of an object to its inertial mass was posed by Sir Isaac Newton and to this day remains a topic of theoretical and experimental studies. It emerges practically in every theory of gravitation. It is well known, for example, that when Newton was constructing his theory of gravitation (see Newton, 1687), he was forced, in order to experimentally solve this problem, to carry out a series of measurements of the periods of oscillations of pendulums of the same length but made of different materials. On the basis of these experiments he incorporated into his theory a postulate according to which the force of gravity is proportional to the amount of matter (or mass) in the object irrespective of the composition and/or size of the object. Thus, in modern terms, Newton clearly understood that each object is characterized by two types of mass, namely, the inertial mass as the measure of inertia in the object and the gravitational mass as the measure of gravitational charge.

At present, in the terminology introduced by Bondi, 1957, it is customary in every theory of gravitation to distinguish between three types of mass: the inertial mass m_i, the passive gravitational mass m_p, and the active gravitational mass m_a. The inertial mass characterizes the ability of an object to acquire one or another acceleration under the action of forces of a nongravitational nature; it enters into Newton's second law, $m_\mathrm{i}a^\alpha = F^\alpha$, and is defined by this law. The passive gravitational mass characterizes the effect that gravitational fields have on the object; it is defined by the expression

$$F_\alpha = -m_\mathrm{p}\frac{\partial U}{\partial x^\alpha}.$$

Finally, the active gravitational mass of an object characterizes its ability to generate a gravitational field.

In different theories of gravitation the interrelation of these three masses may be different. In Newtonian mechanics, for instance, Newton's third law requires that

the active and passive gravitational masses be equal, while the fact that the passive gravitational mass is equal to the inertial mass is postulated.

Actually, the question of how these three types of mass relate to each other comes down to the question of how the various interactions contribute to the inertial and gravitational masses of the object. It is well known, for example, that the study of the motion of various elementary particles and ions in electromagnetic fields has revealed that the strong, weak, and electromagnetic interactions contribute equally to the inertial mass of an object, the contribution being equal to the energy of each interaction divided by c^2, with c the speed of light in empty space.

But in principle there is no reason why these interactions should contribute equally to the other two types of mass. In other words, there is nothing that implies that the field responsible for one interaction cannot be a stronger or weaker source of gravitational field, all other things being equal, than all other material fields. In this case the active mass of an object would not be equal to the inertial mass, and the difference would be the greater the higher the fraction of the energy of the given interaction in the total energy of the object. Hence, in the most general case the possibility cannot be excluded that the value of the active mass of the object will differ from that of the inertial mass:

$$m_\mathrm{a} = m_\mathrm{i} + \sum_A \eta_{\mathrm{a}A} \frac{E_A}{c^2},$$

where E_A is the energy of the field responsible for the Ath interaction, and $\eta_{\mathrm{a}A}$ is a dimensionless parameter characterizing the nonequivalence of the contributions of the Ath field to the inertial and active gravitational masses of the object.

By analogy, the possible difference in the action of an external gravitational field on different types of matter can be described by the following formula for the passive gravitational mass of an object:

$$m_\mathrm{p} = m_\mathrm{i} + \sum_A \eta_{\mathrm{p}A} \frac{E_A}{c^2}. \tag{20.1}$$

Thus, only if $\eta_{\mathrm{p}A} = \eta_{\mathrm{a}A} = 0$ will the gravitational properties of different forms of matter be the same; otherwise gravitational interaction ceases to be universal.

Numerous experiments have been staged to establish the relationship between the various types of mass for the same object. Initially the objects involved had laboratory dimensions. The most famous experiments measured gravimetrically the ratio of the passive gravitational mass to the inertial mass and were conducted by Braginsky and Panov, 1971, Eötvös, Pekár, and Fekete, 1922, and Roll, Krotkov, and Dicke, 1964. The main idea of all these experiments can be formulated as follows.

Let us assume that not all the $\eta_{\mathrm{p}A}$ in (20.1) vanish. Then the passive gravitational mass of an object is generally not equal to its inertial mass. The equations of motion of a point-like object in an external homogeneous gravitational field assume the form

$$m_\mathrm{i}\mathbf{a} = \left(m_\mathrm{i} + \sum_A \eta_{\mathrm{p}A} \frac{E_A}{c^2}\right) \mathbf{g}.$$

This implies that the accelerations

$$\mathbf{a} = \left(1 + \sum_A \eta_{\mathrm{p}A} \frac{E_A}{m_\mathrm{i}c^2}\right) \mathbf{g} \tag{20.2}$$

acquired by different objects in a gravitational field will differ, and the difference will be the greater the greater the change in the second term on the right-hand side of Eq. (20.2) brought on by the change in the substance comprising the object. Hence, if two objects with the same inertial mass but of different composition are placed at the ends of the beam of a torsion balance in an external gravitational field, the resulting torsion pendulum will be subjected to a torque $\mathbf{M}=(m_{1p}-m_{2p})\times(\mathbf{r}\times\mathbf{g})$ proportional to the difference in the passive gravitational masses of these objects.

Measuring the value of the torque acting on the pendulum makes it possible to determine this difference at least in principle. However, to ensure greater accuracy in measuring the ratio $(m_{1p}-m_{2p})/m_1$, the setting is somewhat changed in practical experiments so that the torque caused by the difference in the passive gravitational masses of the two objects changes as a result of this modification according to a harmonic law with a frequency equal to the resonance frequency of the pendulum. To this end either the torsion pendulum was rotated in the Earth's gravitational field or use was made of the natural rotation of the Earth (together with the torsion pendulum) about its axis; in the second case the Sun was taken as the source of the external gravitational field.

All experiments of this type have been conducted over the years for a large number of substances and have shown, with a high accuracy, the absence of any dependence of the ratio $\eta=(m_{1p}-m_{2p})/m_1$ on the type of substance. The series of experiments conducted by Eötvös and his group (see Eötvös, Pekár, and Fekete, 1922) between 1890 and 1922 yielded an estimate $|\eta|<5\times10^{-9}$. Later, the groups of Braginsky (see Braginsky and Panov, 1971) and Dicke (see Roll, Krotkov, and Dicke, 1964) established experimentally that $|\eta|<10^{-12}$ and $|\eta|<10^{-11}$, respectively. The accuracy in determining η achieved in these experiments was great enough to establish the fractional contribution of the strong, weak, and electromagnetic interactions to the passive gravitational mass and the inertial mass of a point-like object. Indeed, since the ratios of the energies of the strong, weak, and electromagnetic interactions are different for different substances, $E_{1A}/m_ic^2-E_{2A}/m_ic^2\neq0$ for each of these interactions. Hence, assuming that the upper bound on $|\eta|$ is caused by the nonequal contributions to m_p and m_i of only one of these interactions, we can find the bounds on the η_{pA}.

Such an analysis shows (see Will, 1981) that the strong-interaction energy and the electrostatic energy of the nuclei provide equal contributions to both masses with an accuracy of about one part in 10^{10}, while the energy of the static magnetic field of the nuclei provides equal contributions to both masses with an accuracy of about one part in 10^6. Moreover, the accuracy achieved by Braginsky's group makes it possible to claim that the weak interaction also makes a fairly equal contribution ($\eta_{pw}<10^{-2}$) to the passive gravitational and inertial masses of an object.

Although these results are interpreted as proof of the equality of the inertial and passive gravitational masses, this does not mean that objects with greater dimensions have gravitational and inertial masses that coincide with the same accuracy. For an object of laboratory dimensions the gravitational self-energy, elastic stress energy, and the like are extremely low in comparison with the object's total energy. For one thing, for an object of mass m and characteristic dimension a, the ratio of the gravitational self-energy to the total energy of the object is

$$\frac{Gm^2/a}{mc^2}=\frac{Gm}{c^2a}\simeq\frac{G\rho a^2}{c^2},$$

where ρ is the density of the object.

This ratio is equal, by order of magnitude, to about 10^{-25} for objects of laboratory dimensions. Hence, if the accuracy of measurement is one part in 10^{12}, nothing can be said of how the gravitational self-energy is distributed between the inertial and gravitational masses of the object. Consequently, the results of gravimetric measurements can serve only as proof that the gravitational mass is equal to the inertial mass for an object of laboratory dimensions, that is, an object whose gravitational self-energy, elastic stress energy, and the like are negligible if compared with the total energy of the object. To determine the ratio of the gravitational mass to the inertial mass for an extended object, we must either drastically raise the precision of gravimetric experiments involving objects of laboratory dimensions (which is impossible at the present level of technology) or carry out measurements that involve objects of greater dimension, say planets, for which the ratio of the gravitational self-energy to the total energy is considerably higher than for objects of laboratory dimensions.

But since gravitational measurement of the ratio of the passive gravitational mass of an extended object (a planet) to its inertial mass is impossible, we must look for phenomena in which the difference between these masses will manifest itself. One is the effect of deviation of the motion of the center of mass of an extended object from a geodesic in the Riemann space-time. The first to notice this possibility was Dicke, 1962, who suggested that the ratio of the gravitational mass to the inertial mass for astronomical bodies differ somewhat from unity if the gravitational self-energy of such bodies varies under change of their position in the gravitational field of other bodies. Later this effect was also studied by Dicke, 1969, Nordtvedt, 1968a, and Will, 1971a.

Having in mind further application of the results of his calculations to the system consisting of the Sun and a planet in the solar system, Will, 1971a, demonstrated on the basis of the PPN formalism that the equations of motion of the center of mass of an extended object (a planet) in the gravitational field of a point-like source (the Sun) at rest assume the form

$$m_i a^\alpha = m_a f^\alpha / R^2,$$

where m_i is the inertial mass of the extended object, m_a is the active gravitational mass of the source, a^α are the components of acceleration of the center of mass of the extended object, and R is the distance between the point-like source of gravitational field and the center of mass of the extended object. Assuming that the velocity of a spherically symmetric extended object is zero, Will arrived at the following formula for vector f^α:

$$\frac{f^\alpha}{m_i} = -n^\alpha \left\{1 - \left[4\beta - \gamma - 3 - \alpha_1 - \xi_1 + \alpha_2 - \frac{1}{3}(\alpha_2 + \xi_2 - \xi_1)\right]\Omega\right], \qquad (20.3)$$

where $n^\alpha = R^\alpha / |\mathbf{R}|$, and the specific gravitational self-energy of the extended object meets the condition

$$\Omega = \frac{1}{2m_i} \int \frac{\rho\rho'}{|\mathbf{x} - \mathbf{x}'|} d^3x\, d^3x' \ll 1.$$

Will, 1971a, first defined the passive gravitational mass in accordance with the condition $f^\alpha = -n^\alpha m_p$. Using this definition, he arrived at the conclusion that

$$\frac{m_p}{m_i} = 1 - \left[4\beta - \gamma - 3 - \alpha_1 - \xi_1 + \alpha_2 - \frac{1}{3}(\alpha_2 + \xi_2 - \xi_1)\right]\Omega. \qquad (20.4)$$

Within this approach the presence of post-Newtonian correction terms in (20.4) was interpreted as the result of breakdown in some theories of gravitation at the

post-Newtonian level of the equality between the passive gravitational mass and the inertial mass of an extended object. It was also stated that the equality of the passive gravitational mass and the inertial mass in the post-Newtonian approximation would mean that the center of mass of the extended object moves along a geodesic. Here it is important to note that the motion of a test body occurs, by definition, along a geodesic, which is determined from the principle of least action: the functional $S = \int L\, dt$ takes the least possible value on a geodesic curve. The equation of this curve is given by varying the Lagrangian function

$$L = -m\left(g_{ik}\frac{dx^i}{dt}\frac{dx^k}{dt}\right)^{1/2}$$

over the coordinates of the particle of mass m placed in the gravitational field corresponding to metric g_{ik} in the Riemann space-time.

In conditions of a real experiment it is difficult to determine whether the center of mass of an extended object moves along a geodesic. One approach to this problem has been to determine from experiments the values of all the required post-Newtonian parameters and, using Will's formula (20.4), find the ratio of the passive gravitational mass to the inertial mass of an extended object and establish the pattern of motion of the center of mass of this object in relation to a specific geodesic in the Riemann space-time.

The first to suggest an experiment of this type was Nordtvedt, 1973. By calculating the motion of the Earth-Moon system in the Sun's gravitational field, he suggested the existence of a number of anomalies in the Moon's motion whose observation might make it possible to measure combinations of post-Newtonian parameters. One such anomaly is the polarization of the Moon's orbit in the direction of the Sun with an amplitude $\delta r \simeq \eta l$, where l is a constant of the order of ten meters, and

$$\eta = (4\beta - \gamma - 3 - \xi_1 - \alpha_1 + \alpha_2) - \frac{1}{3}(\xi_2 + \alpha_2 - \xi_1) - \frac{10}{3}\xi_w.$$

To discover this effect, an analysis was made of the data obtained from measuring the Earth-Moon separation by laser ranging. As a result one group of researchers (see Williams *et al.*, 1976) concluded that $\eta = 0 \pm 0.03$, while another found a close result: $\eta = -0.001 \pm 0.015$. Using these estimates and Will's formula (20.4) for the passive mass, a number of researchers concluded that the ratio of the passive gravitational mass of the Earth to its inertial mass should be close to unity: $m_p/m_1 = 1 \pm (1.5 \times 10^{-11})$. Thus, the data from laser ranging of the Moon would seem to suggest (and this was done by Shapiro, Counselman, and King, 1976, and Williams *et al.*, 1976) that in the post-Newtonian approximation the passive gravitational mass of an extended object is equal to the object's inertial mass and that the center of mass moves along a geodesic in the Riemann space-time.

In a later work, Will, 1981, did not assume that the velocity of the extended object is zero and wrote the equations of the motion of the center of mass of the extended object incorporated in a double system in the form

$$m_1 a^\alpha = -m_p \partial^\alpha U + m_1 a_N^\alpha + m_1 a_{\text{self}}^\alpha, \tag{20.5}$$

where the passive gravitational energy m_p of the spherically symmetric extended object is defined, as before, via (20.4), the a_{self}^α are generated by the gravitational self-energy of the extended object and for a totally conservative metric theory are zero, U stands for the Newtonian gravitational potential of the second object in

the system, and the a_N^{α}, named N-body accelerations by Will, contain only one (nonkinematic) parameter characterizing the extended object, its mass, and are independent of the object's inner structure.

However, this definition of passive mass is incorrect. As we will shortly show (see also Denisov, Logunov, Mestvirishvili, and Chugreev, 1985, and Denisov, Logunov, and Chugreev, 1986), if the passive mass defined in this manner coincides with the inertial mass, Eq. (20.5) is not reduced to the equation of geodesic motion of a particle placed in the total gravitational field of two extended objects (say, the Earth and the Sun) at the center of mass of one of these (the Earth).

It, therefore, seems natural to define the passive gravitational mass of one extended object, the Earth, in such a manner that by setting it equal to the inertial mass we ensure that the equation of motion of the center of mass of the Earth coincides with the equation of the geodesic motion in the total gravitational field. Hence, depending on the ratio of the passive gravitational mass to the inertial mass, the center of mass either follows a geodesic or does not.

The novelty of this approach lies in the fact that the equation of motion of the center of mass of an extended object is compared not to the equation of a geodesic in the effective Riemann space-time, whose curvature is created only by the second extended object, but to the equation of a geodesic in the total gravitational field generated by both objects, including the one the motion of the center of mass of which is being investigated.

In this approach the formula for the ratio of the passive gravitational mass to the inertial mass of the Earth in the gravitational field of the Sun assumes the form (see (20.36))

$$\frac{m_{p\oplus}}{m_{i\oplus}}=1+\frac{2}{3}\left(4\beta-3-\frac{18}{5}\xi_w\right)U_{\oplus}(0)+\left(3+\gamma-4\beta+\frac{10}{3}\xi_w\right)\Omega_{\oplus},$$

where $U_{\oplus}(0)=\int \rho_{\oplus}dV/|\mathbf{x}|$ is the Earth's gravitational self-potential at the Earth's center of mass. The difference between this formula and Will's lies in the presence of a term proportional to $U_{\oplus}(0)$. Clearly, for the values of the PPN parameters in the RTG formalism Will's formula implies that in the post-Newtonian approximation the inertial and gravitational masses are equal, while formula (20.36) implies that this is not so. In this chapter we suggest specific physical experiments whose results will enable interpreting the motion pattern for extended objects. For objects of laboratory dimensions, when the gravitational self-potential is extremely low, $m_p = m_i$ in RTG.

20.1 Post-Newtonian Conservation Laws in Metric Theories of Gravitation

Metric theories of gravitation occupy a special place among the different theories of gravitation since, although there can be many different postulates lying at their base, the action of the gravitational field on matter is achieved in them through the metric tensor of the Riemann space-time. Hence, if the metric generated by sources of gravitational field is known, then to calculate the motion of matter within the framework of a metric theory of gravitation there is no need to know in detail the basic equations of this theory.

A common approach to a unified description in the post-Newtonian approximation of metric theories of gravitation is to employ the PPN formalism in which the metric coefficients of the respective Riemann space-time have the form (17.75)-(17.77).

All subsequent calculations will be carried out in an inertial reference frame whose center of mass moves uniformly and rectilinearly. For this the following conservation laws must be valid on the post-Newtonian level of accuracy: the laws of conservation of energy, conservation of momentum, and conservation of angular momentum, with all three applied to matter and gravitational field taken together. Strictly speaking, the laws of conservation exist only in those metric theories of gravitation in which the covariant equation expressing the law of conservation of the material energy-momentum tensor in the Riemann space-time can be represented in the form of a covariant conservation law for the sum of the symmetric energy-momentum tensors for the gravitational field, $t^{ik}_{(g)}$, and matter, $t^{ik}_{(M)}$, in the pseudo-Euclidean space-time (see Logunov and Mestvirishvili, 1984, 1985a, 1985b, 1986b, Vlasov and Logunov, 1984, and Vlasov, Logunov, and Mestvirishvili, 1984):

$$\nabla_i T^{ik} = D_i\,(t^{ik}_{(g)} + t^{ik}_{(M)}) = 0, \tag{20.6}$$

where ∇_i and D_i are the symbols of covariant differentiation with respect to the Riemannian and pseudo-Euclidean metrics. For a perfect fluid we have

$$T^{ik} = \sqrt{-g}\,\{[p + \rho\,(1+\Pi)]\,u^i u^k - p g^{ik}\}. \tag{20.7}$$

Since in the pseudo-Euclidean space-time there exist ten Killing vectors, on the basis of (20.6) we can obtain ten corresponding integral conservation laws (for energy, momentum, and angular momentum).

Within the PPN formalism, however, there can only exist necessary but insufficient conditions for the existence of ten conservation laws. As (20.6) implies, for this we must represent the covariant derivative of the tensor density T^{ik} with respect to metric g_{ik} in the form of the sum of a time derivative and a three-dimensional divergence term ($D_i = \partial/\partial x^i$ in Cartesian coordinates). In this case the quantities under the differentiation signs will be the components of the tensor $t^{ik}_{(g)} + t^{ik}_{(M)}$.

Let us write with post-Newtonian accuracy the components of the material energy-momentum tensor density (20.7):

$$\begin{gathered} T^{00} = \hat{\rho}\left[1 - \frac{1}{2}v_\nu v^\nu + \Pi + U + O(\varepsilon^4)\right], \\ T^{0\alpha} = \hat{\rho} v^\alpha\left[1 - \frac{1}{2}v_\nu v^\nu + \Pi + U + p/\hat{\rho} + O(\varepsilon^4)\right], \\ T^{\alpha\beta} = \hat{\rho} v^\alpha v^\beta - \gamma^{\alpha\beta} p + O(\varepsilon^6), \end{gathered} \tag{20.8}$$

where $\hat{\rho} = \rho\sqrt{-g}\,u^0 = \rho\left(1 + 3\gamma U - \frac{1}{2}v_\nu v^\nu + O(\varepsilon^4)\right)$ is the conserved mass density of the perfect fluid, a quantity that satisfies the continuity equation

$$\nabla_i \rho u^i \equiv \frac{1}{\sqrt{-g}}\left[\frac{\partial}{\partial t}\hat{\rho} + \frac{\partial}{\partial x^\alpha}\hat{\rho} v^\alpha\right] = 0. \tag{20.9}$$

At $k = 0$ the covariant conservation law $\nabla_i T^{ik} = \partial_i T^{ik} + \Gamma^k_{mn} T^{mn} = 0$ assumes the form

$$\frac{\partial}{\partial t}\left[\hat{\rho}\left(1 + \Pi - \frac{1}{2}v_\nu v^\nu + U\right)\right] + \partial_\alpha\left[\hat{\rho} v^\alpha\left(1 + \Pi - \frac{1}{2}v_\nu v^\nu + U + p/\hat{\rho}\right)\right] \\ - \hat{\rho}\frac{\partial U}{\partial t} - 2\hat{\rho} v^\alpha \partial_\alpha U = \hat{\rho} O(\varepsilon^5). \tag{20.10}$$

Let us transform the last two terms on the left-hand side of (20.10). Employing the equations $\partial^\varepsilon \partial_\varepsilon U = 4\pi\rho$ and $\partial^\varepsilon \partial_\varepsilon V^\alpha = -4\pi\rho v^\alpha$, we obtain

$$\hat{\rho}\,\frac{\partial U}{\partial t}+2\hat{\rho}v^\varepsilon\partial_\varepsilon U\equiv\frac{\partial}{\partial t}\left\{a_1\hat{\rho}U+\frac{2a_1-3}{8\pi}\,\partial_\varepsilon U\partial^\varepsilon U\right\}$$
$$+\partial_\beta\left\{\frac{1-a_1}{4\pi}\,\partial^\beta U\,\frac{\partial U}{\partial t}+\frac{a_2}{4\pi}\,U\partial^\beta\,\frac{\partial U}{\partial t}\right.$$
$$\left.+(a_1+a_2)\,\hat{\rho}Uv^\beta+\frac{2-a_1-a_2}{4\pi}\,\partial_\varepsilon U\,[\partial^\varepsilon V^\beta-\partial^\beta V^\varepsilon]\right\},\quad (20.11)$$

with a_1 and a_2 arbitrary numbers. Hence,

$$t^{00}_{(g)}+t^{00}_{(M)}=\hat{\rho}\left[1+\Pi-\frac{1}{2}\,v_\nu v^\nu+(1-a_1)\,U\right]+\frac{3-2a_1}{8\pi}\,\partial_\varepsilon U\partial^\varepsilon U+\hat{\rho}O\,(\varepsilon^4),\quad (20.12)$$

$$t^{0\alpha}_{(g)}+t^{0\alpha}_{(M)}=\hat{\rho}v^\alpha\left[1+\Pi-\frac{1}{2}\,v_\nu v^\nu+(1-a_1-a_2)\,U+p/\hat{\rho}\right]$$
$$+\frac{a_1-1}{4\pi}\,\partial^\alpha U\,\frac{\partial U}{\partial t}+\frac{a_1+a_2-2}{4\pi}\,\partial_\beta U\,(\partial^\beta V^\alpha-\partial^\alpha V^\beta)$$
$$-\frac{a_2}{4\pi}\,U\partial^\alpha\,\frac{\partial U}{\partial t}+\hat{\rho}O\,(\varepsilon^5).\quad (20.13)$$

At $k=\alpha$, after lengthy calculations, the covariant conservation equation is transformed to the following form:

$$\nabla_i T^{i\alpha}=\frac{\partial}{\partial t}\,(t^{\alpha 0}_{(g)}+t^{\alpha 0}_{(M)})+\frac{\partial}{\partial x^\beta}\,\left(t^{\alpha\beta}_{(g)}+t^{\alpha\beta}_{(M)}\right)+S^\alpha,\quad (20.14)$$

where

$$t^{\alpha 0}_{(g)}+t^{\alpha 0}_{(M)}=\hat{\rho}v^\alpha\left[1+\Pi-\frac{1}{2}\,v_\nu v^\nu+(2\gamma+1)\,U+p/\hat{\rho}\right]$$
$$-\frac{\alpha_2}{4\pi}\,w_\beta\partial^\alpha U\partial^\beta U-\frac{1}{8\pi}\,(4\gamma+2+\alpha_1-2\alpha_2+2\xi_1+4\xi_w)\,\partial^\alpha U\,\frac{\partial U}{\partial t}$$
$$+\frac{\alpha_1}{8\pi}\,\partial_\nu U\partial^\nu U w^\alpha+\frac{1}{8\pi}\,(4\gamma+4+\alpha_1)\,\partial_\beta U\,[\partial^\alpha V^\beta-\partial^\beta V^\alpha],\quad (20.15)$$

and vector S^α is defined thus:

$$S^\alpha=\left[\xi_3\hat{\rho}\Pi+(3\xi_4+2\xi_w)\,p-(\xi_2-\xi_w)\,\frac{1}{8\pi}\,\partial_\nu U\partial^\nu U\right.$$
$$\left.-\frac{\alpha_3+\xi_1+2\xi_w}{2}\,\hat{\rho}v_\nu v^\nu-\frac{\xi_1+2\xi_w}{8\pi}\,\partial_\nu\partial^\nu A-\alpha_3\hat{\rho}v^\varepsilon w_\varepsilon\right]\partial^\alpha U.\quad (20.16)$$

The sum $t^{\alpha\beta}_{(g)}+t^{\alpha\beta}_{(M)}$ is symmetric in indices α and β, but since we will not need it in what follows, we do not give it explicitly.

It can be demonstrated that vector S^α cannot be represented in the form of a four-dimensional divergence of a combination of the generalized gravitational potentials and the characteristics of the perfect fluid. Since in metric theories of gravitation containing ten conservation laws vector S^α must be equal to zero, this implies that the PPN parameters in these theories necessarily satisfy the conditions

$$\xi_3=0,\quad 3\xi_4+2\xi_w=0,\quad \xi_2=\xi_w,\quad \xi_1+2\xi_w=0,\quad \alpha_3=0.\quad (20.17)$$

As is known, if we require that the angular momentum be conserved, then the tensor $t^{ik}_{(g)}+t^{ik}_{(M)}$ must be symmetric. Comparing (20.13) and (20.15), we see that

$t^{\alpha 0}_{(g)} + t^{\alpha 0}_{(M)} = t^{0\alpha}_{(g)} + t^{0\alpha}_{(M)}$ only if

$$\alpha_1 = \alpha_2 = 0, \quad a_1 = -2\gamma, \quad a_2 = 0. \tag{20.18}$$

Thus, the requirement that there be post-Newtonian conservation laws leaves only three PPN parameters independent: γ, β, and ξ_w. The other parameters are: $\xi_1 = -2\xi_w$, $\xi_2 = \xi_w$, $\xi_4 = -(2/3)\,\xi_w$, $\alpha_1 = \alpha_2 = \alpha_3 = \xi_3 = 0$. This is what is known as the completely conservative PPN formalism, and within this formalism we will operate.

Note that all these limitations on the values of the PPN parameters were obtained by Lee, Lightman, and Ni, 1974, who used the pseudotensor approach. This approach, however, is physically meaningless; a critical review of it can be found in Denisov and Logunov, 1982d.

20.2 The Equation of Motion of the Center of Mass of an Extended Object

We start the derivation of the equation by defining the radius vector Y^α of the center of mass of the extended object:

$$m_1 Y^\alpha = \int (t^{00}_{(g)} + t^{00}_{(M)})\, X^\alpha\, dV, \tag{20.19}$$

where m_1 is the inertial mass, or

$$m_1 = \int (t^{00}_{(g)} + t^{00}_{(M)})\, dV.$$

For the function $t^{00}_{(g)} + t^{00}_{(M)}$ we take the expression (20.12) transformed to local form:

$$t^{00}_{(g)} + t^{00}_{(M)} = \hat{\rho}\left(1 + \Pi - \frac{1}{2} U - \frac{1}{2} v_\nu v^\nu\right), \tag{20.20}$$

which is easily interpreted in the following manner: $\hat{\rho}$ is the density of the rest mass of the object, $\hat{\rho}\Pi$ and $\hat{\rho}U$ the densities of the internal and potential energies, respectively, and $-(1/2)\,\hat{\rho} v_\nu v^\nu$ the density of the kinetic energy. On the basis of the differential conservation law

$$\frac{\partial}{\partial x^i} (t^{0i}_{(M)} + t^{0i}_{(g)}) = 0$$

we arrive at a uniform and rectilinear motion of the center of mass of a double system:

$$\frac{d}{dt} \frac{m_{1(1)} Y^\alpha_{(1)} + m_{1(2)} Y^\alpha_{(2)}}{m_{1(1)} + m_{1(2)}} = \frac{\int (t^{0\alpha}_{(M)} + t^{0\alpha}_{(g)})\, dV}{\int (t^{00}_{(M)} + t^{00}_{(g)})\, dV} = \text{const}. \tag{20.21}$$

In addition to the inertial mass we introduce the rest mass of the extended object, $M = \int \rho\, dV$, which, in view of the continuity equation, is time independent. As shown in Fock, 1939, 1959, the post-Newtonian variation in the weighting function in the definition of the center of mass for extended objects has no effect on the equations of motion of spherically symmetric extended objects in the lowest order in L/R, which is the ratio of the characteristic size of such objects to their separation. For this reason all calculations involving the equation that describes the motion of the center of mass of an extended object will be based on $\hat{\rho}$ rather than on $t^{00}_{(g)} + t^{00}_{(M)}$.

The post-Newtonian equations of motion of a perfect fluid can be obtained, following Fock, 1939, 1959, by writing the covariant conservation equation $\nabla_i T^{ik}=0$ at $k=\alpha$. In the completely conservative PPN formalism we have

$$\begin{aligned}\hat{\rho}\,\frac{dv^\alpha}{dt}&+\partial_\beta\left[T^{\alpha\beta}-\hat{\rho}v^\alpha v^\beta\left(1+\Pi+U-\frac{1}{2}v_\nu v^\nu\right)-pv^\alpha v^\beta+\gamma^{\alpha\beta}pU\right]\\&+\hat{\rho}\partial^\alpha U-2(\beta+\gamma)\hat{\rho}U\partial^\alpha U+(\gamma-2)p\partial^\alpha U+v^\alpha\frac{\partial p}{\partial t}\\&+\partial^\alpha p\left(\Pi-\frac{1}{2}v_\nu v^\nu+p/\hat{\rho}\right)-\gamma v_\nu v^\nu\hat{\rho}\partial^\alpha U+(2\gamma+1)\hat{\rho}v^\alpha\frac{\partial U}{\partial t}\\&+2(\gamma+1)\hat{\rho}v^\alpha v^\beta\partial_\beta U+\frac{4\gamma+3-2\xi_w}{2}\hat{\rho}\,\frac{\partial V^\alpha}{\partial t}+\frac{1+2\xi_w}{2}\hat{\rho}\,\frac{\partial N^\alpha}{\partial t}\\&+2(\gamma+1)\hat{\rho}v^\beta(\partial_\beta V^\alpha-\partial^\alpha V_\beta)+\hat{\rho}\partial^\alpha\Phi=\hat{\rho}O(\varepsilon^6),\end{aligned}\tag{20.22}$$

where $d/dt=\partial/\partial t+v_\beta\partial^\beta$ is the convective derivative, and

$$\begin{aligned}\Phi=(\gamma+1-\xi_w)\Phi_1&+\xi_w(A-\Phi_w)+(3\gamma+1-2\beta-2\xi_w)\Phi_2\\&+\Phi_3+(3\gamma-2\xi_w)\Phi_4.\end{aligned}\tag{20.23}$$

Let us consider two spherically symmetric objects that occupy volumes $V_\odot$ and $V_\oplus$, those of the Sun and the Earth, and are separated by a distance much greater than the Earth's radius: $R_\oplus/R=O(\varepsilon)$. The conserved density of mass in this case can be written in the form $\hat{\rho}(x,t)=\rho_\oplus(x,t)+\rho_\odot(x,t)$, with $\rho_\oplus$ nonzero within the Earth's volume and $\rho_\odot$ nonzero within the Sun's volume.

We denote the radius vector connecting the centers of mass of the Sun and the Earth by R^α and the unit vector directed along R^α by $n^\alpha\equiv R^\alpha/R$. Let us now integrate the equation of motion (20.22) over the Earth's volume in the geocentric nonrotating comoving (inertial) reference frame and expand each term in this equation in powers of R^{-1} to within $O(R^{-4})$. If we introduce notation for the integrals characterizing the distribution of matter in the extended objects,

$$\begin{gathered}\Omega_{(i)}=\frac{1}{2M_i}\int\frac{\rho_i\rho_i'}{|x-x'|}d^3x',\quad P_{(i)}=\frac{1}{M_i}\int p_i d^3x',\quad i=\oplus,\odot,\\V_\odot^\alpha=\frac{1}{M_\odot}\int\rho_\odot v^\alpha d^3x',\quad \Pi_\odot=\frac{1}{M_\odot}\int\rho_\odot\Pi\,d^3x',\end{gathered}\tag{20.24}$$

we arrive at an equation describing the acceleration of the Earth's center of mass:

$$a_{(1)}^\alpha=-\frac{M_\odot}{R^2}n^\alpha\left[1+\left(3+\gamma-4\beta+\frac{10}{3}\xi_w\right)\Omega_\oplus\right]+\widetilde{N}^\alpha,\tag{20.25}$$

where the post-Newtonian contribution to the Earth's acceleration is

$$\begin{aligned}\widetilde{N}^\alpha=\frac{M_\odot}{R^2}\Big\{n^\alpha\Big[&(\gamma+1)V_\odot^\varepsilon V_{\odot\varepsilon}+\frac{3}{2}(n_\varepsilon V_\odot^\varepsilon)^2+\frac{3}{2}P_\odot-\Pi_\odot\\&-\left(\frac{5}{2}+\gamma-4\beta+\frac{10}{3}\xi_w\right)\Omega_\odot+(2\beta+2\gamma+1)M_\odot/R\\&+2(\beta+\gamma)M_\odot/R\Big]-(2\gamma+1)V_\odot^\alpha V_\odot^\nu n_\nu\Big\},\end{aligned}\tag{20.26}$$

with the Earth assumed to be a point-like object.

The above expression for $a_{(1)}^\alpha$ represents the acceleration of the Earth's center of mass averaged over the short-period internal motion of matter in the two extend-

ed objects. In this case, as shown in Denisov, Logunov, and Mestvirishvili, 1981, the following tensor virial theorems hold true:

$$3P_{\oplus}=\Omega_{\oplus} \quad \text{and} \quad \int \rho_{\odot} v^{\varepsilon} v_{\varepsilon}\, dV = M_{\odot} V_{\odot\varepsilon} V_{\odot}^{\varepsilon} + 3P_{\odot} - \Omega_{\odot}. \tag{20.27}$$

These were used in deriving (20.25).

To answer the question posed at the beginning of this section and concerning the motion of the Earth's center of mass, we must compare this motion with an idealized pattern, namely, with the motion of a test body in a Riemann space-time whose metric is formally equivalent to the metric generated by two moving extended objects. Then the fact that the expression for the acceleration $a_{(1)}^{\alpha}$ of the center of mass of an extended object coincides with that for the acceleration $a_{(0)}^{\alpha}$ of a point-like object would imply that under similar initial conditions the center of mass of the extended object (the Earth) and the point-like object move along the same trajectory and are governed by the same law of motion. Since a point-like object moves, by definition, along a geodesic, the center of mass in this case moves along a geodesic, too. But if the expressions for $a_{(0)}^{\alpha}$ and $a_{(1)}^{\alpha}$ differ in post-Newtonian correction terms, the center of mass of the extended object will not generally move along a geodesic in the Riemann space-time. Such an approach, among other things, makes it possible to allow naturally for the contribution of the gravitational self-field of a given extended object to the space-time curvature.

20.3 The Geodesic Motion Equation

We wish to study the motion of a point-like object in the vicinity of the Earth's center of mass. Let us write the geodesic equations as follows:

$$\frac{du^i}{ds} + \Gamma^i_{mn} u^m u^n = 0, \tag{20.28}$$

where $ds = (g_{ik}\, dx^i\, dx^k)^{1/2}$ is the interval (or the invariant physical separation), and $u^i = dx^i/ds$ is the 4-vector of velocity. At $i = \alpha$ we have

$$\begin{aligned}
a_{(0)}^{\alpha} = \frac{dV_0^{(\alpha)}}{dt} &= \partial^{\alpha} U\,[1-\gamma V_{(0)}^{\varepsilon} V_{(0)\varepsilon} - 2(\beta+\gamma)U] \\
&+ 2(\gamma+1)\, V_{(0)\varepsilon}\,[\partial^{\alpha} V^{\varepsilon} - \partial^{\varepsilon} V^{\alpha}] - \partial^{\alpha}\Phi \\
&- 2(\gamma+1)\, V_{(0)}^{\alpha} V_{(0)}^{\varepsilon} \partial_{\varepsilon} U - (2\gamma+1)\, V_{(0)}^{\alpha} \frac{\partial U}{\partial t} \\
&- \frac{1}{2}(4\gamma+3-2\xi_{w}) \frac{\partial V^{\alpha}}{\partial t} - \frac{1+2\xi_{w}}{2} \frac{\partial N^{\alpha}}{\partial t} + O(\varepsilon^6),
\end{aligned} \tag{20.29}$$

where $V_{(0)}^{\alpha}$ is the velocity of the point-like object with respect to the Earth's center of mass and Φ is given in (20.23).

We expand all the potentials entering into (20.29) in power series in R^{-1} to within $O(R^{-4})$. Collecting like term yields

$$\begin{aligned}
a_{(0)}^{\alpha} = &-\partial^{\alpha} U_{\oplus} + \tilde{a}_{0}^{\alpha} + \frac{M_{\odot}}{R}[(4\beta+2\gamma-1-3\xi_{w})\,\partial^{\alpha} U_{\oplus} - \xi_{w} n_{\mu} n_{\nu} \partial^{\alpha} U_{\oplus}^{\mu\nu}] \\
&+ \frac{M_{\odot}}{R^2}\Big\{ n^{\alpha}\Big[-1+\gamma V_{(0)}^{\varepsilon} V_{(0)\varepsilon} + \Big(2\beta - \frac{3}{2} - 2\xi_{w}\Big) U_{\oplus} \\
&- 2(\gamma+1)\, V_{(0)}^{\varepsilon}\, V_{\odot\varepsilon}\Big] - 2(\gamma+1)\, V_{(0)}^{\alpha} V_{(0)}^{\varepsilon} n_{\varepsilon} \\
&+ (2\gamma+1)\, V_{(0)}^{\alpha} V_{\odot}^{\varepsilon} n_{\varepsilon} + 2(\gamma+1)\, V_{\odot}^{\alpha} V_{(0)}^{\varepsilon} n_{\varepsilon}
\end{aligned}$$

$$
\begin{aligned}
&-\left(2\beta-\frac{3}{2}-2\xi_w\right) n_\varepsilon U_\oplus^{\alpha\varepsilon}-(1-2\gamma-4\beta \\
&+5\xi_w)\, n_\varepsilon r^\varepsilon \partial^\alpha U_\oplus-\xi_w r^\alpha n_\varepsilon \partial^\varepsilon U_\oplus \\
&+\frac{\xi_w}{2}\left(n_\mu \gamma_{\varepsilon\beta}+2 n_\varepsilon \gamma_{\mu\beta}+3 n_\varepsilon n_\mu n_\nu\right)\left(\partial^{\alpha\mu} f^{\varepsilon\beta}-r^\varepsilon r^\beta \partial^{\alpha\mu} U_\oplus\right)\Big\} \\
&+\frac{M_\odot}{R^3}\left(r^\alpha-3 n^\alpha n_\varepsilon r^\varepsilon\right)+\widetilde{N}^\alpha+\frac{M_\odot}{R^4}\Big[\left(\gamma^{\alpha\varepsilon} r^\mu r_\mu-3 r^\alpha r^\varepsilon\right) n_\varepsilon \\
&+\frac{5}{2} n^\alpha n_\beta n_\nu\left(\gamma^{\mu\nu} r_\varepsilon r^\varepsilon-3 r^\mu r^\nu\right)\Big], \qquad (20.30)
\end{aligned}
$$

where r^α is the radius vector of the test body reckoned from the Earth's center of mass, $U_\oplus$ and $U_\oplus^{\alpha\beta}$ are the Earth's gravitational potentials at the point where the point-like object is situated, and

$$
f^{\mu\nu}=\int \frac{\rho y^\mu y^\nu\, d^3 y}{|\mathbf{r}-\mathbf{y}|}=U_\oplus^{\mu\nu}-r^\mu r^\nu U_\oplus .
$$

The post-Newtonian acceleration $\widetilde{a}_0^\alpha$ is caused completely by the Earth's gravitational field:

$$
\begin{aligned}
\widetilde{a}_0^\alpha= & \partial^\alpha U_\oplus\left[\gamma V_{(0)}^\varepsilon V_{(0)\varepsilon}+2(\beta+\gamma) U_\oplus\right] \\
& -\partial^\alpha\left[\left(\gamma+1-\xi_w\right) \Phi_{1\oplus}+\left(3\gamma+1-2\beta+\xi_w\right) \Phi_{2\oplus}+\Phi_{3\oplus}\right. \\
& \left.+\left(3\gamma-2\xi_w\right) \Phi_{4\oplus}\right]-(2\gamma+1) V_{(0)}^\alpha \partial_\beta V_\oplus^\beta \\
& -2(\gamma+1) V_{(0)}^\alpha V_{(0)}^\varepsilon \partial_\varepsilon U_\oplus-2(\gamma+1) V_{(0)\varepsilon}\left(\partial^\varepsilon V_\oplus^\alpha-\partial^\alpha V_\oplus^\varepsilon\right) \\
& -(\gamma+1) \int \frac{\rho_\oplus v^\alpha v^\varepsilon\left(x_\varepsilon-x_\varepsilon^{\prime}\right) d^3 x^{\prime}}{\left|\mathbf{x}-\mathbf{x}^{\prime}\right|^3}-\frac{4\gamma+3-2\xi_w}{2} \int \frac{\rho_\oplus \partial^{\prime\alpha} U_\oplus d^3 x^{\prime}}{\left|\mathbf{x}-\mathbf{x}^{\prime}\right|} \\
& -\frac{1+2\xi_w}{2} \int \frac{\rho_\oplus\left(x^\alpha-x^{\prime\alpha}\right)\left(x^\beta-x^{\prime\beta}\right) \partial_\beta^{\prime} U_\oplus d^3 x^{\prime}}{\left|\mathbf{x}-\mathbf{x}^{\prime}\right|^3} . \qquad (20.31)
\end{aligned}
$$

The symbol $\oplus$ on the potentials Φ_1, Φ_2, Φ_3, Φ_4, and V^α signifies that the corresponding integrand contains the conserved density only of the Earth.

We are interested in the acceleration of a point-like object whose position and velocity coincide with those of the Earth's center of mass. With post-Newtonian accuracy, $V_{(0)}^\alpha \simeq V_\oplus^\alpha + O(\varepsilon^3)$, while all the gradients of the Earth's gravitational self-potentials vanish (see Denisov, Logunov, and Chugreev, 1986). Moreover, $U_\oplus^{\alpha\beta}(0) = -(1/3)\,\gamma^{\alpha\beta} U_\oplus(0)$. Hence, $\widetilde{a}_0^\alpha = 0$, and the expression (20.30) for the acceleration of a point-like object can be written as follows:

$$
a_{(0)}^\alpha=-n^\alpha \frac{M_\odot}{R^2}\left[1-\frac{2}{3}\left(4\beta-3-\frac{18}{3} \xi_w\right) U_\oplus(0)\right]+\widetilde{N}^\alpha . \qquad (20.32)
$$

20.4 The Earth's Passive Gravitational Mass

Comparison of the expression (20.25) for $a_{(1)}^\alpha$ with the expression (20.32) for $a_{(0)}^\alpha$ suggests that the two quantities are not equal and, hence, the Earth's center of mass does not move along a geodesic in an arbitrary completely conservative metric theory of gravitation. We will postpone the discussion of this question until Section 20.5. Here we will only interpret this important fact through the inequality between the Earth's inertial mass and passive gravitational mass.

We multiply Eq. (20.32) by the inertial mass of the point-like object, which according to (20.19) is defined thus:

$$m_{\mathrm{i}(0)}=m_0\left[1-\frac{1}{2}U_{\oplus}(0)-\frac{1}{2}\frac{M_{\odot}}{R}+O(\varepsilon^4)\right],$$

with m_0 the rest mass of the point-like object. The product is

$$m_{\mathrm{i}(0)}a^{\alpha}_{(0)}=-m_{\mathrm{i}(0)}n^{\alpha}\frac{M_{\odot}}{R^2}+m_0\left[\frac{2}{3}\frac{M_{\odot}}{R^2}n^{\alpha}\left(4\beta-3-\frac{18}{5}\xi_w\right)U_{\oplus}(0)+\widetilde{N}^{\alpha}\right]. \quad (20.33)$$

Since a point-like object moves, by definition, along a geodesic, its inertial and passive gravitational masses are equal, with the result that we can rewrite Eq. (20.33) in the form

$$m_{\mathrm{i}(0)}a^{\alpha}_{(0)}=-m_{\mathrm{p}(0)}n^{\alpha}\frac{M_{\odot}}{R^2}+m_0\left[\frac{2}{3}\frac{M_{\odot}}{R^2}n^{\alpha}\left(4\beta-3-\frac{18}{5}\xi_w\right)U_{\oplus}(0)+\widetilde{N}^{\alpha}\right], \quad (20.34)$$

because $m_{\mathrm{p}(0)}=m_{\mathrm{i}(0)}$ in this case. This equation forms a basis for defining the passive gravitational mass of an extended object. If we define it in such a manner that the fact of its equality with the inertial mass transforms the equation of motion of the center of mass into a geodesic motion equation, then for the Earth's gravitational mass we have, with due regard for (20.34), the following equation:

$$a^{\alpha}_{(1)}=-\frac{m_{\mathrm{p}\oplus}}{m_{\mathrm{i}\oplus}}n^{\alpha}\frac{M_{\odot}}{R^2}+\left[\frac{2}{3}\frac{M_{\odot}}{R^2}n^{\alpha}\left(4\beta-3-\frac{18}{5}\xi_w\right)U_{\oplus}(0)+\widetilde{N}^{\alpha}\right], \quad (20.35)$$

with

$$\frac{m_{\mathrm{p}\oplus}}{m_{\mathrm{i}\oplus}}=1+\frac{2}{3}\left(4\beta-3-\frac{18}{5}\xi_w\right)U_{\oplus}(0)+\left(3+\gamma-4\beta+\frac{10}{3}\xi_w\right)\Omega_{\oplus}. \quad (20.36)$$

Thus, we can say that the deviation of the motion of the center of mass from the motion along a geodesic is due entirely to the deviation from unity of the ratio of the Earth's passive gravitational mass to the inertial mass. Let us now give expressions for these masses:

$$\begin{aligned}
m_{\mathrm{i}\oplus}&=m_{\oplus}\left[1+\Pi_{\oplus}-\Omega_{\oplus}-\frac{1}{2}\frac{M_{\odot}}{R}+O(\varepsilon^4)\right],\\
m_{\mathrm{p}\oplus}&=m_{\oplus}\left[1+\Pi_{\oplus}+\left(3+\gamma-4\beta+\frac{10}{3}\xi_w\right)\Omega_{\oplus}\right.\\
&\left.\quad+\frac{2}{3}\left(4\beta-3-\frac{18}{5}\xi_w\right)U_{\oplus}(0)-\frac{1}{2}\frac{M_{\odot}}{R}+O(\varepsilon^4)\right].
\end{aligned} \quad (20.37)$$

Finally, using the numerical values of $U_{\oplus}(0)$ and $\Omega_{\oplus}$ listed in Chugreev, 1985, we arrive at the following formula for the mass ratio:

$$\frac{m_{\mathrm{p}\oplus}}{m_{\mathrm{i}\oplus}}=1+\left(4\beta-3-\frac{18}{5}\xi_w\right)7.6\times10^{-10}+\left(3+\gamma-4\beta+\frac{10}{3}\xi_w\right)4.6\times10^{-10}.$$

Since in RTG $\beta=\gamma=1$ and $\xi_w=0$, in the post-Newtonian approximation the Earth's passive gravitational mass is not equal to the inertial mass:

$$\frac{m_{\mathrm{p}\oplus}}{m_{\mathrm{i}\oplus}}=1+7.6\times10^{-10}.$$

20.5 Deviation of the Motion of the Earth's Center of Mass from the Reference Geodesic

Let us consider the difference in the accelerations of the Earth's center of mass and of a point-like object moving in the vicinity of the center of mass. Equations (20.25) and (20.32) imply that this difference is of the post-Newtonian order of smallness. Hence, the deviation r^α of the point-like object from the Earth's center of mass can be expected to be small: $r^\alpha/l_\oplus = O(\varepsilon)$. This means that for the acceleration of the point-like object we can take its expression at the Earth's center of mass, (20.32), but allow for the Newtonian term $\partial^\alpha U_\oplus$ in it. According to (20.25) and (20.32),

$$a^\alpha_{(0)} - a^\alpha_{(1)} = n^\alpha \frac{M_\odot}{R^2}\left[\frac{2}{3}\left(4\beta-\gamma-\frac{18}{5}\xi_w\right) n^\alpha U_\oplus(0) + \left(3+\gamma-4\beta+\frac{10}{3}\xi_w\right)\Omega_\oplus\right]$$
$$-\partial^\alpha U_\oplus(r) + O\left(10^{-14}\frac{M_\oplus}{l^2_\oplus}\right). \tag{20.38}$$

Since r^α is small, we can employ the following approximate formula for $\partial^\alpha U_\oplus(r)$:

$$\partial^\alpha U_\oplus(r) = \partial^\alpha U_\oplus(0) + \partial^{\alpha\beta} U_\oplus(0)\, r_\beta + O((r/l_\oplus)^3), \tag{20.39}$$

where $\partial^\alpha U_\oplus(0) = 0$, and

$$\partial^{\alpha\beta} U_\oplus(0) \equiv \int \frac{\rho_\oplus}{y^2}(\gamma^{\alpha\beta} + 3y^\alpha y^\beta/y^2)\, d^3y + \frac{4\pi}{3}\rho_\oplus(0)\,\gamma^{\alpha\beta} = \frac{4\pi}{3}\rho_\oplus(0)\,\gamma^{\alpha\beta}.$$

Thus, we arrive at the following equation for the radius vector $r^\alpha = r^\alpha(t)$:

$$\frac{d^2 r^\alpha}{dt^2} + \frac{4\pi}{3}\rho_\oplus(0)\, r^\alpha = b^\alpha + O((r/l_\oplus)^3), \tag{20.40}$$

where the driving force b^α is directly proportional to the deviation from unity of the ratio of the Earth's passive gravitational mass to the inertial mass, (20.36):

$$b^\alpha = \frac{M_\odot}{R^2} n^\alpha \left(\frac{m_{p\oplus}}{m_{i\oplus}} - 1\right). \tag{20.41}$$

The solution to the oscillation equation (20.40) with zero initial conditions can be written as

$$r^\alpha = \frac{b^\alpha}{\omega_0^2}(\cos\omega_0 t - 1), \tag{20.42}$$

where the period of oscillations corresponding to frequency ω_0 is

$$T = \frac{2\pi}{\omega_0} = \sqrt{\frac{3}{\pi\rho_\oplus(0)}} \simeq 55 \text{ min}.$$

Solution (20.42) implies that the plane in which these oscillations occur is fixed by vector $\mathbf{n}$, which slowly rotates with a frequency corresponding to a period of one year.

The amplitude of the oscillations is

$$A = \frac{|b^\alpha|}{\omega_0^2}$$
$$= \frac{3M_\odot}{4\pi R^2\rho_\oplus(0)}\left[\frac{2}{3}\left(4\beta-3-\frac{18}{5}\xi_w\right)U_\oplus(0) + \left(3+\gamma-4\beta+\frac{10}{3}\xi_w\right)\Omega_\oplus\right]$$
$$\simeq 1.2\times 10^{-4}\left(4\beta-3-\frac{18}{5}\xi_w\right)\text{cm} + 7.5\times 10^{-5}\left(3+\gamma-4\beta+\frac{10}{3}\xi_w\right)\text{cm}. \tag{20.43}$$

Although this value of the amplitude constitutes a small quantity, its post-Newtonian order of smallness is $\varepsilon^3 l_\oplus$, with the result that this effect can be detected in experiments that possess post-Newtonian accuracy. We also note that the oscillation amplitude is independent of the Earth's rotation.

20.6 The Law of Motion of an Electrically Charged Test Body

Experimentally, the above-discussed effect can be observed only indirectly, for instance, by studying the motion of a test body near the Earth's surface with the aim of determining the spectral composition of the relativistic Earth tides and isolating the waves that must be observed if we assume that the Earth's center of mass moves along a geodesic.

Due to the smallness of the post-Newtonian correction terms in the acceleration of gravity, gathering experimental data requires a long time. Hence, the test body must be at rest in relation to the laboratory, which means we must compensate for the acceleration of gravity by applying a force of nongravitational nature to the body. Following Nordtvedt, 1971, for such a counterforce we take the electric force acting on an electrically charged test body. The value of the electric field strength will be directly proportional to the force of gravity acting on the test charge and will be expressed in terms of nonelectromagnetic quantities, with the result that our calculations hold true for ordinary gravimetric instruments.

Thus, the electric counterforce is an auxiliary element and its use is justified only because we can then easily calculate the post-Newtonian corrections which it introduces into the calculation of the components of the acceleration of gravity, which are directly measurable by gravimeters and tilted pendulums. Here, as we will shortly see on the assumption that the Earth's center of mass moves along a geodesic in the total gravitational field of the Earth and the Sun, the amplitudes of some relativistic tidal waves change. Experimental measurement of the magnitude of such waves gives an idea of how the Earth's center of mass moves.

Let us take the law of motion of a test charge e of mass m in the combined gravitational field of the Earth and the Sun and in an electromagnetic field, characterized by a 4-potential A^i, created artificially on the surface of the planet. The Lagrangian function is well known for this motion:

$$L = -m\left(g_{ik}\frac{dx^i}{dt}\frac{dx^k}{dt}\right)^{1/2} - e g_{ik} A^i \frac{dx^k}{dt} - \frac{1}{16\pi}\int \sqrt{-g}\, F_{ik}F^{ik}\, dV, \qquad (20.44)$$

where $F_{ik} = \nabla_i A_k - \nabla_k A_i$ is the electromagnetic field-strength tensor.

Varying the action that corresponds to the Lagrangian function (20.44), we arrive at the Maxwell field equations in the Riemann space-time:

$$\nabla_i F^{ik} = 4\pi j^k. \qquad (20.45)$$

If, for the sake of simplicity, we assume that the electromagnetic current j^k is generated by moving point charges e_a, that is,

$$j^k = \sum_a \frac{e_a \delta(\mathbf{r}-\mathbf{r}_a)}{\sqrt{-g}}\,\frac{dx^k}{dt}, \qquad (20.46)$$

with $\delta(\mathbf{r}-\mathbf{r}_a)$ the common Euclidean delta function and $\mathbf{r}_a$ the radius vector of the ath charge, then Eq. (20.45) for $k=0$ assumes the form

$$\nabla^2 A^0 = -\frac{\partial^2}{\partial t^2}A^0 + 4\pi\sum_a e_a\delta(\mathbf{r}-\mathbf{r}_a)\,[1+(1-\gamma)U] + (3-\gamma)\,\partial^\alpha A^0 \partial_\alpha U + O(A^0\varepsilon^4), \qquad (20.47)$$

with $\nabla^2 \equiv \partial_\alpha \partial^\alpha$. The solution to this equation can be written with post-Newtonian accuracy as follows:

$$A^0 = \sum_a \frac{e_a}{|\mathbf{r}-\mathbf{r}_a|} + \frac{1}{2}\frac{\partial^2}{\partial t^2}\sum_a e_a |\mathbf{r}-\mathbf{r}_a| - \frac{1+\gamma}{2}\sum_a \frac{e_a U(\mathbf{r}_a)}{|\mathbf{r}-\mathbf{r}_a|}$$
$$+ \frac{\gamma-3}{2}\left[\sum_a \int \frac{\rho'}{|\mathbf{r}-\mathbf{r}'|}\frac{e_a d^3r'}{|\mathbf{r}'-\mathbf{r}_a|} - U(\mathbf{r})\sum_a \frac{e_a}{|\mathbf{r}-\mathbf{r}_a|}\right]. \tag{20.48}$$

We use this solution to find the expression for the law of motion of an electric test charge in the electromagnetic and gravitational fields. Varying the Lagrangian function (20.44) over the coordinates of the charge leads us to the following equation of motion:

$$u^k \nabla_k u^i \equiv \frac{du^i}{ds} + \Gamma^i_{pm} u^p u^m = \frac{e}{m} F^{ik} u_k. \tag{20.49}$$

For $i = \alpha$ this equation transforms into

$$a^\alpha = \frac{d^2 r^\alpha}{dt^2} = \dot{a}^\alpha_{(0)} + a^\alpha_{\text{em}}, \tag{20.50}$$

where r^α is the radius vector of the test charge reckoned from the Earth's center of mass, and $a^\alpha_{(0)}$ is the contribution to the acceleration caused by gravitational interaction and is defined in (20.30). The contribution to the acceleration caused by the Lorentz force and its post-Newtonian corrections, which we have denoted by a^α_{em}, is given by the following formula:

$$a^\alpha_{\text{em}} = \frac{e}{m}\left\{\partial^\alpha A^0\left[1-(2\gamma+3)U + \frac{1}{2}V^\varepsilon_{(0)}V_{(0)\varepsilon}\right] - 2\partial^\alpha U A_0\right.$$
$$\left. + \partial^\alpha A^\varepsilon V_{(0)\varepsilon} - \partial_\varepsilon A^\alpha V^\varepsilon_{(0)} - \frac{\partial}{\partial t}A^\alpha\right\}. \tag{20.51}$$

In this formula it is sufficient to employ the following expression for the vector potential A^α:

$$A^\alpha = \sum_a \frac{e_a V^\alpha_a}{|\mathbf{r}-\mathbf{r}_a|},$$

where V^α_a is the velocity of the test charge in the geocentric nonrotating comoving reference frame in which all calculations will be carried out.

Selecting the charge and mass of the test body so that the electric force compensates for the acceleration of gravity g and restricting the accuracy of calculations by 10^{-14} (the sensitivity of the current crop of gravimeters is 10^{-12}), we obtain

$$a^\alpha_{\text{em}} = \frac{e}{m}\{E^\alpha[1-(2+3\gamma)U + V^\varepsilon_{(0)}V_{(0)\varepsilon}] - 2V^\alpha_{(0)}V^\varepsilon_{(0)}E_\varepsilon + \delta E^\alpha\}, \tag{20.52}$$

where we have introduced the following notation for the field acting on the test charge:

$$E^\alpha = \sum_a e_a \frac{r^\alpha - r^\alpha_a}{|\mathbf{r}-\mathbf{r}_a|^3}, \quad \delta E^\alpha = -\frac{3}{2}\sum_a e_a \frac{(r^\alpha - r^\alpha_a)[V_{(0)\varepsilon}(r^\varepsilon - r^\varepsilon_a)]^2}{|\mathbf{r}-\mathbf{r}_a|^5}.$$

Here we have allowed for the fact that, within the chosen accuracy, $U(\mathbf{r}_a) = U(\mathbf{r})$ and $V_a^\alpha = V_{(0)}^\alpha$. The Newtonian potential U at the surface of the Earth is defined thus:

$$U = \frac{M_\oplus}{r_\oplus} + \frac{M_\odot}{R} + \frac{M_\odot}{R^2} n_\varepsilon r_\oplus^\varepsilon . \tag{20.53}$$

20.7 Transformation to Physical Coordinates

Time t and coordinates r^α, in terms of which the above relations have been written, are coordinate quantities. They are related to the physical observables τ and l^α through the following formulas:

$$d\mathbf{l}^2 = -\left(g_{\alpha\beta} - \frac{g_{0\alpha} g_{0\beta}}{g_{00}}\right) dr^\alpha\, dr^\beta, \tag{20.54}$$

$$d\tau = \sqrt{g_{00}}\, dt + \frac{g_{0\alpha}}{\sqrt{g_{00}}}\, dr^\alpha. \tag{20.55}$$

In the post-Newtonian approximation these formulas become simpler:

$$d\mathbf{l}^2 = -g_{\alpha\beta}\, dr^\alpha\, dr^\beta\, [1 + O(\varepsilon^6)], \tag{20.56}$$

$$d\tau = \sqrt{-g_{00}}\, dt\, [1 + O(\varepsilon^4)]. \tag{20.57}$$

Nordtvedt, 1971, used (20.53) to integrate Eq. (20.56) and found that (R is assumed constant)

$$l^\alpha = \left(1 + \gamma \frac{M_\odot}{R}\right) r^\alpha - \frac{\gamma}{2} \frac{M_\odot}{R^2} n^\alpha r_\varepsilon r^\varepsilon + \gamma \frac{M_\odot}{R^2} r^\alpha r^\varepsilon n_\varepsilon. \tag{20.58}$$

We will use this formula later. Note, however, that the absence in it of a term responsible for the contribution of the Earth's gravitational field is due to the fact that the corresponding post-Newtonian corrections to the acceleration of gravity are time independent and, therefore, need not be taken into account.

The corrections to r^α in (20.58) have a post-Newtonian order of smallness. Hence, to write the equation of motion of the charge in physical coordinates it is sufficient to transform into these coordinates only the gradient of the Newtonian potential, $\partial^\alpha U_\oplus$, and the electric field E^α. The quantity $\rho_\oplus(\mathbf{r})\, d^3r$ is the element of the Earth's rest mass, with the result that $\rho_\oplus(\mathbf{r})\, d^3r = \rho_\oplus(\mathbf{l})\, d^3l$. Combining this fact with (20.58) yields

$$\begin{aligned} \partial^\alpha U_\oplus(\mathbf{r}) &= \partial^\alpha U_\oplus(\mathbf{l}) \left(1 + 2\gamma \frac{M_\odot}{R}\right) - \frac{\gamma}{2} n^\alpha \frac{M_\odot}{R^2} l_\varepsilon \partial^\varepsilon U_\oplus \\ &\quad + \frac{\gamma}{2} \frac{M_\odot}{R^2} l^\alpha n_\varepsilon \partial^\varepsilon U_\oplus - \frac{\gamma}{6} \frac{J_\oplus}{l^3} \frac{l_\oplus^2 M_\odot}{R^2} (\gamma^{\alpha\beta} + 3 m^\alpha m^\beta) n_\beta + O(10^{-15} g), \end{aligned} \tag{20.59}$$

where $m^\alpha = l^\alpha / l$ is a unit normal to the surface of the Earth, and $J_\oplus = M_\oplus^{-1} l_\oplus^{-2} \int \rho_\oplus r^2 d^3r$ is the moment of inertia of the Earth per unit mass. Similarly, for the electric field the transformation is

$$\begin{aligned} E^\alpha(\mathbf{r}) = \sum_a e_a \frac{r^\alpha - r_a^\alpha}{|\mathbf{r} - \mathbf{r}_a|^3} &= E^\alpha(\mathbf{l}) \left(1 + 2\gamma \frac{M_\odot}{R} + 2\gamma \frac{M_\odot}{R^2} n_\varepsilon l^\varepsilon\right) \\ &\quad + \gamma \frac{M_\oplus}{R^2} (n^\alpha l^\varepsilon E_\varepsilon - l^\alpha n^\varepsilon E_\varepsilon) + O(10^{-15} g). \end{aligned} \tag{20.60}$$

Note that formula (20.59) coincides with the result of Nordtvedt, 1971 (arrived at in another manner), while in the electric field transformation formula obtained by Nordtvedt there is no last term of (20.60).

The correction terms that emerge as a result of the transformation to invariant time have the form

$$\frac{d^2l^\alpha}{d\tau^2} = (1+2U)\frac{d^2l^\alpha}{dt^2} + V^\alpha_{(0)}\partial_\beta V^\beta + V^\alpha_{(0)}V^\beta_{(0)}\partial_\beta U. \tag{20.61}$$

20.8 A Formula for the Strength of the Compensating Electric Field

Let us now write the condition for the compensation of the acceleration of gravity in an inertial reference with respect to which the Earth rotates with an angular velocity $\mathbf{\Omega}$:

$$\frac{d^2l^\alpha}{d\tau^2} = (\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{l}))^\alpha. \tag{20.62}$$

This equation describes the rotation of vector $\mathbf{l}$ of constant length about vector $\mathbf{\Omega}$ with a constant angular velocity $|\,\mathbf{\Omega}\,|$. Let us find the second derivative of l^α with respect to t. Using the definitions $dr^\alpha/dt = V^\alpha_{(0)} - V^\alpha_\oplus$ and $dR^\alpha/dt = V^\alpha_\oplus - V^\alpha_\odot$ and the order-of-magnitude estimates $M_\odot/R = O\,(10^{-8})$, $r/R = O\,(10^{-5})$, and $\mathbf{n}\cdot(\mathbf{V}_\oplus - \mathbf{V}_\odot) \simeq O\,(e_\oplus V_\odot) = O\,(10^{-6})$, with $e_\oplus = 0.167$ the eccentricity of the ecliptic, we obtain

$$\frac{d^2l^\alpha}{dt^2} = \left(\frac{dV^\alpha_{(0)}}{dt} - \frac{dV^\alpha_\oplus}{dt}\right)\left(1+\gamma\frac{M_\odot}{R}\right) + O\,(10^{-15}g). \tag{20.63}$$

Before we write the final expression for the strength of the compensating electric field, several remarks are in order.

First, let us go over to the positive scalar product of two vectors, $\mathbf{a}\cdot\mathbf{b} = -a_\varepsilon b^\varepsilon \geqslant 0$.

Second, in view of the spherical symmetry in the distribution of matter in the Earth, we employ the following expansions of the gravitational potentials and their gradients for $|\,\mathbf{l}\,| \geqslant l_\oplus$:

$$\begin{aligned} \partial^\alpha U_\oplus &= M_\oplus l^\alpha/l^3, \\ U^{\alpha\beta}_\oplus &= \frac{M_\oplus}{l}\left[m^\alpha m^\beta - \frac{J_\oplus}{3}\frac{l^2_\oplus}{l^2}(\gamma^{\alpha\beta} + 3m^\alpha m^\beta)\right], \\ \partial^\alpha U^{\varepsilon\beta} &= \frac{M_\oplus}{l^2}\Big[3m^\alpha m^\beta m^\varepsilon + \gamma^{\alpha\beta}m^\varepsilon + \gamma^{\alpha\varepsilon}m^\beta \\ &\quad - J_\oplus\frac{l^2_\oplus}{l^2}(m^\alpha\gamma^{\varepsilon\beta} + m^\beta\gamma^{\alpha\varepsilon} + m^\varepsilon\gamma^{\alpha\beta} + 5m^\alpha m^\beta m^\varepsilon)\Big]. \end{aligned} \tag{20.64}$$

Third, we employ in our discussion the Newtonian value of the electric field strength:

$$\frac{e}{m}E^\alpha = \partial^\alpha U_\oplus + (\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{l}))^\alpha. \tag{20.65}$$

Fourth, we note that Will, 1981, has analyzed the R^{-1}-post-Newtonian contribution to the acceleration of gravity. Using the data obtained by Warburton and Goodkind, 1976, (see also Melchior, 1978) through the employment of a superconducting gravimeter, he established the upper limit on the PPN parameter ξ_w:

$$|\xi_w| < 10^{-3}. \tag{20.66}$$

Hence, in R^{-2}-post-Newtonian terms this parameter can be set equal to zero, since if we allow for this limit the corresponding terms will be smaller than the limiting value of $10^{-14}g$.

Fifth, we introduce a new notation for the Earth's orbital velocity in the heliocentric reference frame ($V^\alpha_\odot$ is the Sun's velocity in the geocentric reference frame): $V^\alpha_\odot = v^\alpha$.

With all these remarks in mind, we can write the final formula for the compensating electric field strength in the following form ($l = l_\oplus$):

$$\begin{aligned}
\frac{e}{m} E^\alpha_{\rm ng} = {} & \frac{M_\oplus m^\alpha}{l^2_\oplus} + (\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{l}))^\alpha \\
& - \frac{M_\odot M_\oplus}{R l^2_\oplus} \{ m^\alpha [4\beta - \gamma - 3 - 3\xi_w (1 + J_\oplus)] \\
& \qquad + 2\xi_w n^\alpha \mathbf{m}\cdot\mathbf{n} (1 - J_\oplus) - m^\alpha (\mathbf{m}\cdot\mathbf{n})^2 \xi_w (2 - 5J_\oplus) \} \\
& + \frac{M_\oplus}{R^2} \Big\{ (3 + \gamma - 4\beta) \Big[\Omega_\oplus n^\alpha + \frac{1}{2} \frac{M_\oplus}{l_\oplus} \Big(n^\alpha - m^\alpha \mathbf{m}\cdot\mathbf{n} - \frac{J_\oplus}{3} (n^\alpha - 3m^\alpha \mathbf{m}\cdot\mathbf{n}) \Big) \Big] \\
& \qquad + 2(\gamma + 1) \, [v^\alpha \mathbf{n}\cdot(\mathbf{\Omega} \times \mathbf{l}) - n^\alpha (\mathbf{\Omega} \times \mathbf{l})\cdot\mathbf{v}] \Big\} \\
& + \frac{M_\odot l_\oplus}{R^3} (m^\alpha - 3n^\alpha \mathbf{m}\cdot\mathbf{n}) \\
& - \frac{3}{2} \frac{M_\odot l^2_\oplus}{R^4} [n^\alpha + 2m^\alpha \mathbf{m}\cdot\mathbf{n} - 5m^\alpha (\mathbf{m}\cdot\mathbf{n})^2] + O(10^{-15}g).
\end{aligned} \tag{20.67}$$

Note that the last two groups of terms are the Newtonian quadrupole ($\propto R^{-3}$) and octupole ($\propto R^{-4}$) tidal accelerations. The subscript "ng" on the electric field strength signifies that the Earth's center of mass moves not along a geodesic and the acceleration of the center of mass is described via (20.25).

Let us now see how the electric field strength (directly proportional to the acceleration measured by gravimetric instruments) changes if we assume that the Earth's center of mass does move along a geodesic. For this we only need to substitute $dV^\alpha_{(0)}(0)/dt$, the acceleration of a point-like object at the Earth's center of mass described via (20.32), for acceleration $dV^\alpha_\oplus/dt$ ($\equiv a^\alpha_{(1)}$). This changes the field strength in comparison to $E^\alpha_{\rm ng}$, as can easily be deduced from (20.25) and (20.32), by the following quantity:

$$\begin{aligned}
\frac{e}{m} E^\alpha_{\rm ng} - \frac{e}{m} E^\alpha_{\rm g} = - \frac{M_\odot n^\alpha}{R^2} \Big\{ & \frac{2}{3} \Big(4\beta - 3 - \frac{18}{5} \xi_w \Big) U_\oplus(0) \\
& + \Big(3 + \gamma - 4\beta + \frac{10}{3} \xi_w \Big) \Omega_\oplus \Big\}.
\end{aligned} \tag{20.68}$$

This difference is equal to the driving force in the oscillation equation (20.40) and proportional to the deviation from unity of the ratio of the Earth's passive gravitational mass to the inertial mass.

On the basis of (20.67) and (20.68) we can arrive at a formula for $E^\alpha_{\rm g}$ by performing the following substitution in the formula for $E^\alpha_{\rm ng}$:

$$(3 + \gamma - 4\beta)\, \Omega_\oplus \to \frac{2}{3} (3 - 4\beta)\, U_\oplus(0). \tag{20.69}$$

20.9 Studying the Motion of the Earth's Center of Mass by Gravimetric Experiments

As is known (see Melchior, 1978), the vertical projection of the acceleration of gravity is measured by gravimeters and the horizontal projections, by tilted (horizontal) pendulums.

The vertical projection of E^{α}_{ng} is given by the following formula:

$$\begin{aligned}\frac{e}{m}\mathbf{E}_{\text{ng}}\cdot\mathbf{m} = {} & \frac{M_{\oplus}}{l^2_{\oplus}} + l_{\oplus}[(\Omega\cdot\mathbf{m})^2 - \Omega^2] \\ & - \frac{M_{\odot}M_{\oplus}}{Rl^3_{\oplus}}[4\beta - \gamma - 3 - 3\xi_{\omega}(1+J_{\oplus}) - (\mathbf{m}\cdot\mathbf{n})^2\xi_{\omega}(1-3J_{\oplus})] \\ & + \frac{M_{\odot}}{R^2}\Big\{(3+\gamma-4\beta)\,\mathbf{m}\cdot\mathbf{n}\left(\Omega_{\oplus} + \frac{1}{3}\frac{M_{\oplus}J_{\oplus}}{l_{\oplus}}\right) \\ & \qquad + 2(\gamma+1)\,[(\mathbf{m}\cdot\mathbf{v})\,\mathbf{n}\cdot(\Omega\times\mathbf{l}) - (\mathbf{m}\cdot\mathbf{n})(\Omega\times\mathbf{l})\cdot\mathbf{v}]\Big\} \\ & + \frac{M_{\odot}l_{\oplus}}{R^3}[1 - 3(\mathbf{m}\cdot\mathbf{n})^2] \\ & - \frac{3}{2}\frac{M_{\odot}l^2_{\oplus}}{R^4}(\mathbf{m}\cdot\mathbf{n})[3 - 5(\mathbf{m}\cdot\mathbf{n})^2] + O(10^{-15}g). \end{aligned} \tag{20.70}$$

Here we are interested only in the spectrum of the post-Newtonian term proportional to $\Omega_{\oplus}$ since absolute measurement in a gravimetric experiment of the harmonics corresponding to this term will enable us to answer the question of whether the Earth's center of mass moves along a geodesic.

We denote the sidereal frequency of the Earth's rotation about its axis (i.e. rotation considered with respect to fixed stars) by Ω and the sidereal frequency of the Earth's revolution about the Sun, by ω. If we assume that at $\tau = 0$ the Earth is at perihelion, then in the geocentric ecliptic coordinates, a system generally accepted in astronomy, the following expressions (see Duboshin, 1975) for vectors $\mathbf{n}$, $\mathbf{m}$, and $\mathbf{v}$ and the quantity R^{-1} hold true:

$$\begin{aligned} \mathbf{n} = {} & -\left[\cos\omega\tau + e_{\oplus}(\cos 2\omega\tau - 1) + \frac{3}{8}e^2_{\oplus}(3\cos 3\omega\tau + \cos\omega\tau - 4)\right]\mathbf{e}_x \\ & - \left[\sin\omega\tau + e_{\oplus}\sin 2\omega\tau + \frac{3}{8}e^2_{\oplus}(3\sin 3\omega\tau + \sin\omega\tau)\right]\mathbf{e}_y + O(e^3_{\oplus}), \\ \mathbf{m} = {} & \cos\theta\cos(\Omega\tau - \varphi)\,\mathbf{e}_x + [\cos\theta\cos\alpha\sin(\Omega\tau - \varphi) + \sin\theta\sin\alpha]\,\mathbf{e}_y \\ & + [\sin\theta\cos\alpha - \cos\theta\sin\alpha\,\sin(\Omega\tau - \varphi)]\,\mathbf{e}_z, \\ \mathbf{v} = {} & (\sin\omega\tau\,\mathbf{e}_x - \cos\omega\tau\,\mathbf{e}_y)(M_{\odot}/p)^{1/2}[1 + O(e_{\oplus})], \\ R^{-1} = {} & p^{-1}[1 + e_{\oplus}\cos\omega\tau + e^2_{\oplus}(\cos 2\omega\tau - 1) + O(e^3_{\oplus})], \end{aligned} \tag{20.71}$$

where θ and φ are the geographic latitude and longitude of the point where the gravimeter is positioned, and $\alpha = 23°27'$ is the obliquity of the ecliptic. The unit base vector $\mathbf{e}_x$ points to the perihelion and lies in the ecliptic, the vector $\mathbf{e}_z$ is orthogonal to this plane, and $\mathbf{e}_y$ is normal to $\mathbf{e}_x$ and $\mathbf{e}_z$. The astronomical unit p is equal to 1.496×10^{13} cm.

From (20.70) it follows that the spectral composition of the $\Omega_{\oplus}$-term is determined by the scalar product $\mathbf{m}\cdot\mathbf{n}$, since within the assumed degree of

accuracy we can neglect the annual variations in R and assume that $R = p$. Substituting the expansions from (20.71), we obtain

$$\mathbf{m}\cdot\mathbf{n} = -\frac{(1-\cos\alpha)\cos\theta}{2}\cos[(\Omega+\omega)\tau-\varphi] - \frac{\cos\theta}{2}(1+\cos\alpha)\cos[(\Omega-\omega)\tau-\varphi] - \sin\theta\sin\alpha\sin\omega\tau + O(e_{\oplus}). \tag{20.72}$$

Since $(1+\cos\alpha)/(1-\cos\alpha) = 23.1$, in what follows we consider only the $(\Omega-\omega)$-wave. Its amplitude is maximal at $\theta = 0°$, that is, at the equator. In this case, using (20.71), we arrive at the following expression for the Newtonian wave at this frequency (in the theory of tidal waves denoted by $(S_1)_\theta$):

$$(S_1)_{\theta=0°} = \frac{3}{64}\frac{M_\odot l_\oplus^3}{p^4}(1+\cos\alpha)(23-10\cos\alpha-25\cos^2\alpha)\cos[(\Omega-\omega)\tau-\varphi] = -0.65\cos[(\Omega-\omega)\tau-\varphi]\ \text{nGal}. \tag{20.73}$$

Here we use the unit of nanoGal to measure the amplitudes of tidal waves: $1\ \text{nGal} = 10^{-9}$ Gal, $1\ \text{Gal} = 1\ \text{cm/s}^2$, $g = 982.04$ Gal.

It is characteristic that the Newtonian quadrupole $(\Omega-\omega)$-wave $(\propto p^{-3})$, which generally has an amplitude of about 10^4 nGal, is zero at the equator, while the octupole wave (20.73) is equal, in order of magnitude, to the relativistic $(\Omega-\omega)$-wave, whose amplitude is

$$A^{\text{ng}}_{\theta=0°}(\Omega-\omega) = (3+\gamma-4\beta)\frac{M_\odot}{p^2}\left(\Omega_\oplus + \frac{M_\oplus}{3l_\oplus}J_\oplus\right)\frac{1+\cos\alpha}{2}\cos[(\Omega-\omega)\tau-\varphi] = 0.14(3+\gamma-4\beta)\cos[(\Omega-\omega)\tau-\varphi]\ \text{nGal}. \tag{20.74}$$

If the Earth's center of mass were to travel along a geodesic, then, according to (20.69), the gravimeter would have measured the following wave instead of (20.74):

$$A^{\text{g}}_{\theta=0°} = -\frac{M_\odot}{p^2}\left[\frac{2}{3}(4\beta-3)U_\oplus(0) - \frac{1}{3}(3+\gamma-4\beta)\frac{M_\oplus}{l_\oplus}J_\oplus\right] \times \frac{1+\cos\alpha}{2}\cos[(\Omega-\omega)\tau-\varphi] = -[0.48(4\beta-3)+0.14(3+\gamma-4\beta)]\cos[(\Omega-\omega)\tau-\varphi]\ \text{nGal}. \tag{20.75}$$

Note that for $\gamma = \beta = 1$ the wave $A^{\text{ng}}_{\theta=0°}(\Omega-\omega)$ vanishes in RTG, while $A^{\text{g}}_{\theta=0°}$ has an amplitude of 0.48 nGal.

If in a gravimetric experiment no non-Newtonian additional term to wave $(S_1)_{\theta=0°}$ is discovered, this is proof that the center of mass does not move along a geodesic. Note also that no choice of latitude can make the relativistic ω-wave commensurable with the Newtonian ω-wave. For this reason we do not consider the relativistic ω-wave, just as we do not consider the $(\Omega+\omega)$-harmonic.

20.10 Studying the Motion of the Earth's Center of Mass by Tiltmeters

Using horizontal pendulums and tiltmeters, we can measure the variations in the tangential components of the acceleration of gravity (see Melchior, 1978). At present there are many types of horizontal pendulums and tiltmeters (Backer, 1984, and Melchior, 1978).

Let us project the general expression (20.67) for the electric field strength onto the unit vector $\boldsymbol{\sigma}$ tangential to the surface of the Earth:

$$\begin{aligned}\frac{e}{m}\mathbf{E}_{\rm ng}\cdot\boldsymbol{\sigma} &= (\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{l}))\cdot\boldsymbol{\sigma} - 2\xi_w\frac{M_\oplus}{l^2}\frac{M_\odot}{R}(\mathbf{m}\cdot\mathbf{n})(\mathbf{n}\cdot\boldsymbol{\sigma})(1-J_\oplus)\\ &+\frac{M_\odot}{R^2}\Big\{(\mathbf{n}\cdot\boldsymbol{\sigma})(3+\gamma-4\beta)\Big[\Omega_\oplus+\frac{1}{2}\frac{M_\odot}{l_\oplus}\Big(1-\frac{J_\oplus}{3}\Big)\Big]\\ &\quad+2(\gamma+1)\left[(\mathbf{v}\cdot\boldsymbol{\sigma})(\mathbf{n}\cdot(\boldsymbol{\Omega}\times\mathbf{l}))-(\mathbf{n}\cdot\boldsymbol{\sigma})((\boldsymbol{\Omega}\times\mathbf{l})\cdot\mathbf{v})\right]\Big\}\\ &-3(\mathbf{n}\cdot\boldsymbol{\sigma})(\mathbf{m}\cdot\mathbf{n})\frac{M_\odot l_\oplus}{R^3}-\frac{3}{2}(\mathbf{n}\cdot\boldsymbol{\sigma})(1-5(\mathbf{m}\cdot\mathbf{n})^2)\frac{M_\odot l_\oplus^2}{R^4}\\ &+O(10^{-15}g). \end{aligned}\tag{20.76}$$

The projection of the electric field strength for the case where the Earth's center of mass travels along a geodesic can be obtained by introducing the substitution (20.69). Let us write vector $\boldsymbol{\sigma}$ in the selected coordinate system:

$$\begin{aligned}\boldsymbol{\sigma} = &-[\sin\rho\sin(\Omega\tau-\varphi)+\sin\vartheta\cos\rho\cos(\Omega\tau-\varphi)]\,\mathbf{e}_x\\ &+[\cos\alpha(\sin\rho\cos(\Omega\tau-\varphi)-\sin\vartheta\cos\rho\sin(\Omega\tau-\varphi))\\ &\quad+\sin\alpha\cos\vartheta\cos\rho]\,\mathbf{e}_y\\ &+[\cos\alpha\cos\vartheta\cos\rho-\sin\alpha(\sin\rho\cos(\Omega\tau-\varphi)\\ &\quad-\sin\vartheta\cos\rho\sin(\Omega\tau-\varphi))]\,\mathbf{e}_z.\end{aligned}\tag{20.77}$$

By direct substitution we can see that $\boldsymbol{\sigma}^2 = 1$ and $\boldsymbol{\sigma}\cdot\mathbf{m} = 0$. Vector $\boldsymbol{\sigma}$ is characterized by an azimuthal angle ρ such that $\rho = 0°$ when $\boldsymbol{\sigma}$ points north.

The spectral composition of the terms characterizing the motion of the Earth's center of mass is specified, as (20.76) implies, by the scalar product $\mathbf{n}\cdot\boldsymbol{\sigma}$:

$$\begin{aligned}\mathbf{n}\cdot\boldsymbol{\sigma} &= \frac{1-\cos\alpha}{2}\sqrt{\sin^2\rho+\cos^2\rho\sin^2\vartheta}\cos[(\Omega+\omega)\tau-\delta-\varphi]\\ &+\frac{1+\cos\alpha}{2}\sqrt{\sin^2\rho+\cos^2\rho\sin^2\vartheta}\cos[(\Omega-\omega)\tau-\delta-\varphi]\\ &-\sin\alpha\cos\vartheta\cos\rho\sin\omega\tau+O(e_\oplus),\end{aligned}\tag{20.78}$$

where $\delta = \tan^{-1}(\tan\rho/\sin\vartheta)$. As in Section 20.9, we do not consider the $(\Omega+\omega)$-component.

By selecting two parameters, latitude and azimuth, we must find experimental conditions in which the background Newtonian wave is minimal and the relativistic one is maximal. Employing (20.71) and (20.78), we can easily show that at $\vartheta = 0°$ and $\rho = \pm 90°$ the Newtonian quadrupole $(\Omega-\omega)$-wave ($\propto p^{-3}$) vanishes, while the octupole tidal $(\Omega-\omega)$-wave ($\propto p^{-4}$) is

$$\begin{aligned}(S_t)_{\vartheta=0°,\rho=90°} &= \frac{3}{32}\frac{M_\odot l_\oplus^2}{p^4}[5\cos^2\alpha-30\cos\alpha+21]\sin[(\Omega-\omega)\tau-\varphi]\\ &= -0.23\sin[(\Omega-\omega)\tau-\varphi]\ \text{nGal},\end{aligned}\tag{20.79}$$

and the relativistic $(\Omega-\omega)$-wave is

$$\begin{aligned}A^{\rm ng}_{\vartheta=0°,\rho=90°} &= \frac{M_\odot}{2p^2}(3+\gamma-4\beta)(1+\cos\alpha)\Big[\Omega_\oplus+\frac{1}{2}\frac{M_\oplus}{l_\oplus}\Big(1-\frac{J_\oplus}{3}\Big)\Big]\\ &\quad\times\sin[(\Omega-\omega)\tau-\varphi]\\ &= 0.93(3+\gamma-4\beta)\sin[(\Omega-\omega)\tau-\varphi]\ \text{nGal}.\end{aligned}\tag{20.80}$$

For $\gamma = \beta = 1$ the last wave vanishes and, hence, it is theoretically predicted that there is no deviation from the Newtonian value (20.79).

If we were to assume that the Earth's center of mass moves along a geodesic, then instead of (20.80) we would arrive at the following expression for the relativistic $(\Omega - \omega)$-wave:

$$A^{g}_{\vartheta=0^\circ,\rho=90^\circ} = \frac{M_{\odot}}{6p^2}(1+\cos\alpha)\left[\frac{M_{\oplus}}{2l_{\oplus}}(3-J_{\oplus})(3+\gamma-4\beta)+2(3-4\beta)U_{\oplus}(0)\right]$$
$$\times \sin[(\Omega-\omega)\tau-\varphi]$$
$$= [0.48(3-4\beta)+0.14(3+\gamma-4\beta)]\sin[(\Omega-\omega)\tau-\varphi]\,\text{nGal}. \quad (20.81)$$

At $\gamma = \beta = 1$ the amplitude of this wave is 0.48 nGal, which is twice the Newtonian amplitude.

If as a result of experiments no such harmonic is discovered, it can be stated that the Earth's center of mass does not move along a geodesic.

In conclusion let us give an expression for annual waves at the equator, where there is no Newtonian quadrupole wave and the quadrupole wave (S_a) is commensurable with the relativistic wave:

$$A^{ng}_{\vartheta=\rho=0^\circ}(\omega) = -\frac{M_{\odot}}{p^2}(3+\gamma-4\beta)\left[\Omega_{\oplus}+\frac{M_{\oplus}}{2l_{\oplus}}\left(1-\frac{J_{\oplus}}{3}\right)\right]\sin\alpha\sin\omega\tau$$
$$= -0.41(3+\gamma-4\beta)\sin\omega\tau\,\text{nGal}, \quad (20.82)$$

$$(S_a)_{\vartheta=\rho=0^\circ} = \frac{3M_{\odot}l_{\oplus}}{16p^4}\sin\alpha(12-5\sin^2\alpha)\sin\omega\tau = 2.11\sin\omega\tau\,\text{nGal}, \quad (20.83)$$

$$A^{g}_{\vartheta=\rho=0^\circ}(\omega) = \frac{M_{\odot}}{3p^2}\sin\alpha\left[\frac{M_{\odot}}{2l_{\oplus}}(3+\gamma-4\beta)(3-J_{\oplus})+2(3-4\beta)U_{\oplus}(0)\right]\sin\omega\tau$$
$$= [0.21(3-4\beta)-0.07(4\beta-\gamma-3)]\sin\omega\tau\,\text{nGal}. \quad (20.84)$$

At $\gamma = \beta = 1$ the amplitudes of these waves are, respectively,

$$\bar{A}^{ng} = 0, \quad \bar{S}_a = -0.41\ \text{nGal}, \quad \bar{A}^{g} = -0.21\ \text{nGal}.$$

If the wave $\bar{A}^{g}$ is absent from such measurements, we can say that the Earth's center of mass does not travel along a geodesic.

It must be emphasized that the relativistic tidal waves on the Earth considered in this section will become detectable by measurements when the present accuracy of gravimetric and tiltmeter measurements has increased by a factor of 50 to 100.

Aside from the variations in the acceleration of gravity at the Earth's surface, the fact that the Earth's center of mass does not travel along a geodesic may be studied by investigating the post-Newtonian corrections to the separation between the Earth and an artificial Earth satellite.

20.11 Studying the Motion of the Earth's Center of Mass in an Experiment Involving an Artificial Earth Satellite

As in gravimetric studies of the acceleration of a test body that is at rest with respect to a terrestrial laboratory, an artificial Earth satellite may be considered, in the first approximation, to remain at a fixed (in time) distance from the center of mass of the planet. Extending this analogy, we can say that in the case at hand the centrifugal force plays the role of a counterforce compensating for the Earth's gravity. The difference here lies in the fact that the test body in a gravimeter does

not move along a geodesic because a counterforce of a nongravitational origin acts on it, while the motion of a drift-free artificial Earth satellite can be described by a geodesic equation. Moreover, in gravimetric experiments the directly measurable quantities are the components of the acceleration, while in experiments involving artificial Earth satellites the measurable quantity is the Earth-satellite separation. Derivation of the experimentally measurable quantities in this case generally follows the pattern just considered.

First let us note that as a result of transferring from the coordinate variables t and $\mathbf{r}$ to the physical variables τ and $\mathbf{l}$ the acceleration of the satellites becomes

$$\frac{d^2l^\alpha}{d\tau^2} = \left(\frac{dV^\alpha_{(0)}}{dt} - \frac{dV^\alpha_\oplus}{dt}\right)\left(1+\gamma\frac{M_\oplus}{R}\right) - 2\gamma\frac{M_\odot}{R^2}V^\alpha_{(0)}(\mathbf{V}_{(0)}\cdot\mathbf{n})$$
$$+ 2\gamma\frac{M_\odot M_\oplus}{R^2 l} m^\alpha(\mathbf{m}\cdot\mathbf{n}) + O(10^{-15}g). \tag{20.85}$$

Here we have allowed for the fact that with an accuracy sufficient for our purposes $\mathbf{V}_{(0)}\cdot\mathbf{m} = 0$ and $\mathbf{V}^2_{(0)} = M_\oplus/l$ for an artificial Earth satellite traveling along a circular orbit of radius l with a velocity $\mathbf{V}_{(0)}$ (the reference frame is the same as before).

We will need the following combination of gradients of gravitational potentials, which can easily be calculated if we employ the fact that the Earth is spherically symmetric:

$$\frac{1}{2}(n_\mu\gamma_{\varepsilon\beta} + 2n_\varepsilon\gamma_{\mu\beta} + 3n_\mu n_\varepsilon n_\beta)(\partial^{\alpha\mu}f^{\varepsilon\beta} - l^\varepsilon l^\beta\partial^{\alpha\mu}U_\oplus)$$
$$= \frac{M_\oplus}{l}\left\{-\frac{9}{2}m^\alpha(\mathbf{m}\cdot\mathbf{n})^3\left(1-\frac{7}{3}j_\oplus\frac{l^4_\oplus}{l^4}\right) + \frac{3}{2}n^\alpha(\mathbf{m}\cdot\mathbf{n})^2\left(3j_\oplus\frac{l^4_\oplus}{l^4}-1\right)\right.$$
$$+ m^\alpha(\mathbf{m}\cdot\mathbf{n})\left(\frac{7}{2} - J_\oplus\frac{l^2_\oplus}{l^2} - \frac{9}{2}j_\oplus\frac{l^4_\oplus}{l^4}\right)$$
$$\left.+ n^\alpha\left(\frac{1}{2} - \frac{9}{10}j_\oplus\frac{l^4_\oplus}{l^4} - \frac{1}{3}J_\oplus\frac{l^2_\oplus}{l^2}\right)\right\}, \tag{20.86}$$

where $j_\oplus = M^{-1}_\oplus l^{-4}_\oplus \int \rho_\oplus x^4 dV$ is the reduced fourth-order energy moment. Substituting into (20.85) the expression (20.30) for $dV^\alpha_{(0)}/dt$ and $dV^\alpha_\oplus/dt$ and employing the transformation formula (20.59) for a Newtonian potential, we find that the satellite acceleration in the geocentric inertial reference frame is expressed thus:

$$\frac{d^2l^\alpha}{d\tau^2} = -\frac{M_\oplus l^\alpha}{l^3} + \Delta^\alpha + \frac{M_\oplus M_\odot}{Rl^2}[m^\alpha(4\beta-\gamma-3-3\xi_w) + 2\xi_w n^\alpha(\mathbf{m}\cdot\mathbf{n})$$
$$-3m^\alpha(\mathbf{m}\cdot\mathbf{n})^2]$$
$$+\frac{M_\odot}{R^2}\left\{-n^\alpha\Omega_\oplus\left(3+\gamma-4\beta+\frac{10}{3}\xi_w\right)\right.$$
$$+\frac{M_\oplus}{l}\left[\frac{1}{2}n^\alpha(4\beta-3-\gamma-3\xi_w-3\xi_w(\mathbf{m}\cdot\mathbf{n})^2)\right.$$
$$\left.-m^\alpha(\mathbf{m}\cdot\mathbf{n})\left(2\beta-\frac{\gamma}{2}+\frac{5}{2}+\frac{9}{2}\xi_w(\mathbf{m}\cdot\mathbf{n})^2-\frac{11}{2}\xi_w\right)\right]$$
$$\left.+2V^\alpha_{(0)}(\mathbf{n}\cdot\mathbf{V}_{(0)}) + 2(\gamma+1)(n^\alpha(\mathbf{V}_{(0)}\cdot\mathbf{v}) - v^\alpha(\mathbf{n}\cdot\mathbf{V}_{(0)}))\right\}$$
$$-\frac{M_\odot}{R^3}(l^\alpha - 3n^\alpha(\mathbf{l}\cdot\mathbf{n})) + \frac{3}{2}\frac{M_\odot l^2}{R^4}[2m^\alpha(\mathbf{m}\cdot\mathbf{n}) + n^\alpha - 5n^\alpha(\mathbf{m}\cdot\mathbf{n})^2]$$
$$+O(10^{-15}g). \tag{20.87}$$

The post-Newtonian corrections, which we denote by Δ^α, are due solely to the Earth's gravitational field proper. Since these terms are time independent (in a reference frame that rotates together with the Earth), they will be of no interest to us. It must be emphasized that the terms discarded in (20.87) and proportional to second- and fourth-order moments of inertia provide a contribution exceeding $10^{-15}g$ only for satellites in near-Earth orbits.

Formula (20.87) gives the acceleration of an artificial Earth satellite with respect to the Earth's center of mass, whose motion in turn is specified by Eq. (20.25). If we assume that the Earth's center of mass travels along a geodesic in the total gravitational field of the Sun and the Earth, then in (20.87) we must introduce the substitution (20.69). Let us now calculate the post-Newtonian corrections to the Earth-satellite separation.

We assume in the first approximation that the satellite is moving along a circular orbit of a constant radius l; the plane of the orbit is fixed by the direction tn which the conserved (specific) angular momentum $\mathbf{L} = \mathbf{l} \times (d\mathbf{l}/d\tau)$ points. In ihe second approximation we have

$$l(t) = l + \varepsilon(\tau), \quad \mathbf{L}(\tau) = \mathbf{L} + \mathbf{x}(\tau). \tag{20.88}$$

Allowing for the fact that the force of gravity is compensated for by a centrifugal barrier, or that

$$M_\oplus/l^2 = L^2/l^3, \tag{20.89}$$

we find that

$$\frac{d^2\varepsilon}{d\tau^2} + \Omega^2\varepsilon = 2\,\frac{\mathbf{L}\cdot\mathbf{x}}{l^3} + \boldsymbol{\delta}\cdot\mathbf{m}, \quad \frac{d\mathbf{x}}{d\tau} = \mathbf{l}\times\boldsymbol{\delta}, \tag{20.90}$$

where the oscillation frequency Ω coincides with the orbital revolution frequency of the satellite, $\Omega = (M_\oplus/l^3)^{1/2}$, and $\boldsymbol{\delta}$ stands for all post-Newtonian and tidal corrections to the acceleration.

If the projections of the above quantities vary according to the harmonic law, or

$$\begin{gathered} \boldsymbol{\delta}\cdot\mathbf{m} = A\cos\tilde{\omega}\tau + B\sin\tilde{\omega}\tau, \\ (\mathbf{m}\times\boldsymbol{\delta})\cdot\mathbf{L}/L = C\cos\tilde{\omega}\tau + D\sin\tilde{\omega}\tau, \end{gathered} \tag{20.91}$$

we find that

$$\varepsilon(\tau) = \frac{A - 2(\Omega/\tilde{\omega})D}{\Omega^2 - \tilde{\omega}^2}\cos\tilde{\omega}\tau + \frac{B + 2(\Omega/\tilde{\omega})C}{\Omega^2 - \tilde{\omega}^2}\sin\tilde{\omega}\tau. \tag{20.92}$$

The geostationary orbit is most suitable for observing a satellite. In such an orbit the satellite appears to hang stationary over a point on the Earth's equator at an altitude of approximately $h = 3.6 \times 10^4$ km. In this case, the radar observation of the satellite is not restricted by the time of observation. The distance to the satellite is determined from the time it takes the pulses to travel by multiplying this time by the speed of light, since, as shown by Baierlein, 1967, all relativistic effects associated with deflection and delay of the beam in the Earth's gravitational field can be ignored.

Let us now consider the harmonic composition of the scalar products $\mathbf{m}\cdot\mathbf{n}$ and $\mathbf{L}\cdot(\mathbf{m}\times\mathbf{n})$, since these products determine the Earth-satellite separation harmonics to which the terms containing $\Omega_\oplus$ and $U_\oplus(0)$ contribute. Employing (20.71), we find that

$$\begin{gathered} \mathbf{m}\cdot\mathbf{n} = -\frac{1+\cos\alpha}{2}\cos(\Omega-\omega)\tau - \frac{1-\cos\alpha}{2}\cos(\Omega+\omega)\tau, \\ (\mathbf{m}\times\mathbf{n})\cdot\mathbf{L}/L = \frac{1+\cos\alpha}{2}\sin(\Omega-\omega)\tau + \frac{1-\cos\alpha}{2}\sin(\Omega+\omega)\tau, \end{gathered} \tag{20.93}$$

with Ω, ω, and α defined earlier when we discussed the motion of the Earth's center of mass in gravimetric experiments.

Let us consider the $(\Omega-\omega)$-harmonic of the Earth-satellite separation. According to (20.87) and (20.91)-(20.93) we have

$$\varepsilon^{ng}(\tau)=\frac{M_\odot}{4R^3}\frac{1+\cos\alpha}{\omega\Omega}\left\{3\left(3+\gamma-4\beta+\frac{10}{3}\xi_w\right)\Omega_\oplus\right.$$
$$\left.+\frac{M_\oplus}{l}\left(3+\gamma-4\beta+\frac{\xi_w}{16}(39\cos^2\alpha-42\cos\alpha+127)\right)\right\}$$
$$\times[1+O(\omega/\Omega)]\cos(\Omega-\omega)\tau+\varepsilon_0(\tau). \tag{20.94}$$

In the case of geodesic motion the expression for the $(\Omega-\omega)$-harmonic of $\varepsilon^g(\tau)$ is obtained by introducing the substitution (20.69).

The term denoted in (20.94) as $\varepsilon_0(t)$ is the Newtonian contribution to $\varepsilon(\tau)$:

$$\varepsilon_0(\tau)=\frac{15}{128}\frac{M_\odot l^2}{R^4\Omega\omega}(1+\cos\alpha)(3\cos^2\alpha+14\cos\alpha-13)$$
$$\times[1+O(\omega/\Omega)]\cos(\Omega-\omega)\tau$$
$$=16.4\cos(\Omega-\omega)\tau\text{ m}. \tag{20.95}$$

In RTG $\gamma=\beta=1$ and $\xi_w=0$, whereby for the nongeodesic motion of the center of mass the separation ε^{ng} is equal to the Newtonian value, or $\varepsilon^{ng}=\varepsilon_0$, while if the center of mass travels along a geodesic (with the same values of the PPN parameters), we have

$$\varepsilon^g=15.92\cos(\Omega-\omega)\tau\text{ m}.$$

If as a result of statistical processing of the data on the Earth-satellite separation no such half-meter additional term is discovered, this fact may serve as an indirect indication that the Earth's center of mass does not move along a geodesic.

The modern level of experimental technique (see Vessot, 1984) provides hope that the satellite experiment suggested here will soon be realized.

20.12 Effects Associated with the Presence of a Preferred Reference Frame

A theory of gravitation in which at least one of the parameters α_1, α_2, or α_3 is nonzero possesses a preferred reference frame. Predictions of such theories of gravitation concerning standard effects will coincide with the results of observation only if the solar system is chosen as the preferred reference frame. However, it is more reasonable to assume that the solar system, which moves with respect to other stellar systems, is no different from such systems and, therefore, cannot be taken as the preferred universal rest frame for such theories. Since a preferred rest frame must in some way be distinguishable from other reference frames, it is reasonable to link this system with the center of mass of the Galaxy or even the universe. In this case the solar system will be moving with respect to the preferred rest frame with a speed of about $10^{-3}c$, which is of the same order of magnitude as the orbital speed of the solar system with respect to the center of mass of the Galaxy. This enables the observation of a number of effects associated with the motion relative to the preferred rest frame (see Misner, Thorne, and Wheeler, 1973), which makes it possible to estimate the parameters α_1, α_2, and α_3.

In theories of gravitation with a preferred rest frame the gravitational constant G, measured in gravitational experiments, will depend on the Earth's motion

in relation to such a rest frame. For the fractional change $\Delta G/G$ we have (Misner, Thorne, and Wheeler, 1973)

$$\frac{\Delta G}{G}=\left(\frac{\alpha_2}{2}+\alpha_3-\alpha_1\right)\mathbf{w}\cdot\mathbf{v}$$
$$+\frac{1}{4}\alpha_2\left[(\mathbf{v}\cdot\mathbf{e}_r)^2+2(\mathbf{w}\cdot\mathbf{e}_r)(\mathbf{v}\cdot\mathbf{e}_r)+(\mathbf{w}\cdot\mathbf{e}_r)^2\right],$$

where $\mathbf{v}$ is the Earth's orbital velocity, $\mathbf{w}$ the velocity of the Sun with respect to the preferred rest frame, and $\mathbf{e}_r$ the unit vector directed from the gravimeter to the Earth's center.

Because the Earth rotates about its axis, vector $\mathbf{e}_r$ changes its orientation with respect to vectors $\mathbf{v}$ and $\mathbf{w}$, which results in a periodic variation of the scalar products $\mathbf{v}\cdot\mathbf{e}_r$ and $\mathbf{w}\cdot\mathbf{e}_r$ with a period of approximately 12 hours. This leads to respective periodic variations in the value of acceleration of gravity, namely, for a point of observation at the latitude θ we have

$$\frac{\Delta g}{g}\simeq 3\alpha_2 10^{-8}\cos^2\theta.$$

By analyzing the results of gravimetric experiments, Will, 1971, 1981, found that the fractional change in g does not exceed 10^{-11}, or $|\Delta g|/g<10^{-11}$. This yields the following bound on α_2:

$$|\alpha_2|<4\times10^{-4}.$$

The motion of the Earth around the Sun also leads to periodic variations in $\mathbf{w}\cdot\mathbf{v}$ with a period of about one year. These variations result in the Earth being stretched and compressed which, in turn, leads to periodic variations in the angular velocity of the Earth's rotation due to the variations in the Earth's moment of inertia,

$$\frac{\Delta\omega}{\omega}=3\times10^{-9}\left(\alpha_3+\frac{2}{3}\alpha_2-\alpha_1\right).$$

From the results of observations it follows that

$$\left|\alpha_3+\frac{2}{3}\alpha_2-\alpha_1\right|<0.02.$$

The motion of the solar system with respect to the universe's center can lead to an anomalous shift in the perihelions of the planets. For Mercury (Will, 1981), the additional contribution to the perihelion shift (in seconds of arc per century) is

$$\delta\varphi_0=-123\alpha_1+92\alpha_2+1.4\times10^5\alpha_3+63\xi_w.$$

Comparison with observed results and unification of all the bounds on the α's yield

$$|\xi_w|<2\times10^{-3},\ |\alpha_1|<10^{-3},\ |\alpha_2|<4\times10^{-4},\ |\alpha_3|<2\times10^{-7}.$$

In RTG $\alpha_1=\alpha_2=\alpha_3=0$, and, therefore, the described effects are not present.

20.13 Effects Associated with Anisotropy with Respect to the Center of Mass of the Galaxy

In the theories of gravitation in which parameter ξ_w is nonzero there can be anisotropy effects associated with the gravitational field of the Galaxy (Will, 1973).

If we assume that the mass of the Galaxy is concentrated at its center at a distance R from the solar system, the gravitational field generated by this mass will

lead to periodic variations in the reading of a gravimeter with a period of 12 hours:

$$\frac{\Delta G}{G} = \xi_w \left(1 - \frac{3K}{mr^3}\right) \frac{M}{R} (\mathbf{e}_r \cdot \mathbf{e}_R)^2,$$

where K, m, and r are the moment of inertia, the mass, and the radius of the Earth, respectively, $\mathbf{e}_r$ is the unit vector directed from the gravimeter to the Earth's center, and $\mathbf{e}_R = \mathbf{R}/R$.

Another effect of this kind is the anomalous perihelion shift of planets, a shift caused by the anisotropy associated with the Galaxy:

$$\delta\varphi_0 = \frac{\pi\xi_w}{2} \frac{M}{R} \cos^2\beta \cos^2(\omega - \lambda),$$

where λ and β are the angular coordinates of the center of the Galaxy, and ω is the perihelion angle of a planet in geocentric coordinates.

Comparison with the results of observations yields the following upper bound on ξ_w:

$$|\xi_w| < 10^{-3}.$$

In RTG $\xi_w = 0$, and, therefore, all the effects associated with the anisotropy caused by the gravitational field of the Galaxy are absent.

Chapter 21. The Peters-Mathews Coefficients in RTG

As is well known, in addition to GR other variants of the theory of gravitational interaction are being actively discussed in the literature. At present there are roughly two avenues along which research of gravitation theories is being done: One examines the extent to which the theories obey general theoretical requirements, namely, the completeness of a theory and its self-consistency, the covariance of the basic equations and additional conditions, an analysis of the solution to the problem of the energy-momentum of a gravitational field, and other similar aspects. This work has made it possible to narrow the range of theories requiring further investigation and, at the same time, brought to light the logical contradictions inherent in the theoretical scheme of GR from the standpoint of physics, primarily, the absence of conservation laws for matter and gravitational field taken together.

The second avenue analyzes the extent to which the predictions put forward by various theories agree with the results of gravitational experiments and seeks experimental situations in which different theories must provide different predictions. The interest in these aspects has grown considerably since experimenters achieved post-Newtonian accuracy and theorists built the PPN formalism, the basic theoretical tool for an analysis of post-Newtonian effects. These studies have further narrowed the range of viable theories of gravitation that claim to correctly describe physical reality.

At present, in connection with the discovery of the binary pulsar system PSR 1913 + 16 and the possible existence of similar systems, the literature contains studies of the possibilities of employing the results of observation of such systems as a new experimental test for various theories of gravitation. There are two main reasons why researchers are interested in these systems. First, in the case of a compact binary system containing a pulsar, there is the possibility, statistically analyzing the radiation emitted by the pulsar, of determining the parameters of the orbit of each component in the system with a high degree of accuracy. Second, the characteristics of binary compact systems (the component masses being roughly equal to the Sun's mass, the size of the orbits on the order of the Sun's radius, the

revolution periods small, and the eccentricity sufficiently great) make these systems the most favorable objects for observing a number of fine gravitational effects. For one thing, they offer the possibility of indirectly measuring the energy lost by such systems to gravitational waves.

In addition, we may also expect that direct detection of gravitational waves emitted by binary systems will become possible. Then establishing the radiation pattern and spectral characteristics of gravitational radiation will further broaden employment of the results of observation of such systems as a crucial test for the majority of theories of gravitation.

To analyze the emissive power of compact binary system in different theories of gravitation and compare the results of observations, Will, 1977, suggested using the following general expression for the energy lost by a binary system to gravitational waves:

$$-\frac{dE}{dt}=\frac{8m^2M^2}{15R^4}\left[k_1v^2-k_2\left(\frac{R_\alpha v^\alpha}{R}\right)^2+\frac{5}{8}k_d(\Omega_1-\Omega_2)^2\right], \tag{21.1}$$

where k_1 and k_2 are the Peters-Mathews coefficients, k_d the dipole radiation coefficient, m and M the reduced and total masses of the system, R the separation between the components, v and $R_\alpha v^\alpha/R$ the total and radial relative velocities of the components, and Ω_1 and Ω_2 are defined, as usual, via (20.24).

In such an approach, to each theory of gravitation there corresponds a definite set of values of coefficients k_1, k_2, and k_d that characterizes the theory in the weak gravitational wave approximation to the same extent as the set of post-Newtonian parameters characterizes the post-Newtonian limit of this theory. By comparing the values in this set with the values found in experiments it will be possible to establish a correspondence between the prediction of each theory of gravitation and the results of observation.

As (21.1) implies, the energy lost by a binary system to gravitational waves is, generally, not a positive quantity: for $k_1 < k_2$ or $k_d < 0$ the right-hand side of (21.1) may become negative for some binary systems. Hence, in theories of gravitation in which $k_1 < k_2$ or $k_d < 0$ there may be an emission of gravitational waves carrying negative energy, which is physically meaningless. Hence, such theories must be excluded.

After the values of the coefficients k_1, k_2, and k_d are found from the results of observation of binary systems of the PSR 1913+16 type, the restrictions imposed on theories of gravitation become more rigorous. Let us find the value of these coefficients for RTG. To this end we consider the case of two neutron stars moving along an orbit in the gravitational field generated by these stars. We wish to calculate the energy lost by such a system to gravitational waves. In accordance with the model commonly used in such cases, we assume that both stars are spherically symmetric and static. In addition, we assume that the gravitational fields generated by these stars have a strength that enables us to use the post-Newtonian formalism in determining the motions in the system. Since in compact binary systems the value of the gravitational self-potential at the surface of an object, U_{self}, is considerably higher than the potential of gravitational interaction, U_{int}, we will assume that $\Omega \simeq U_{\text{self}} \simeq \varepsilon$ and $U_{\text{int}} \simeq \varepsilon^2$. This means that the ratio of the characteristic dimension L of each star to the separation R between them must coincide, in order of magnitude, with ε, or $L/R \sim \varepsilon$.

The components (15.34) of the gravitational field can conveniently be written as

$$\Phi^{\alpha\beta}=-\frac{2}{r}\left[P^\alpha_\nu P^\beta_\mu-\frac{1}{2}P^{\alpha\beta}P_{\mu\nu}\right]E^{\mu\nu}, \tag{21.2}$$

where r is the distance from the center of mass of the binary system to the observation point,

$$E^{\mu\nu} = \int dV \left(X^\nu X^\mu - \frac{1}{3}\gamma^{\mu\nu}X_\alpha X^\alpha\right)[q]_{\text{ret}}, \tag{21.3}$$

with

$$q = T^{00} + 2n_\alpha T^{0\alpha} + n_\alpha n_\beta T^{\alpha\beta}, \quad n^\alpha = X^\alpha/|\mathbf{X}|. \tag{21.4}$$

Let us find the post-Newtonian expansion of q. If for the energy-momentum tensor we take expansion (19.16), we get

$$q = \hat{\rho}\left[1 + \frac{1}{2}v^2 + \Pi + U + p/\hat{\rho} + 2n_\nu v^\nu + (n_\nu v^\nu)^2 + O(\varepsilon^3)\right]. \tag{21.5}$$

Since (21.3) contains the retarded value of (21.5), we must, allowing for the estimate $v \simeq \varepsilon$, expand $[q]_{\text{ret}}$ in the neighborhood of the retarded time $t' = t - r$:

$$[q]_{\text{ret}} = q(t') - \dot{q}(t')\, n_\nu X^\nu + \frac{1}{2}\ddot{q}(t')(n_\nu X^\nu)^2 + \rho O(\varepsilon^3),$$

where X^ν is the radius vector of the points of the objects in the reference frame connected with the center of mass of the binary system. From (21.5) and (19.19) it follows that

$$\begin{aligned} [q]_{\text{ret}} = {} & \hat{\rho}\left[1 + \frac{1}{2}v^2 + \Pi + U + p/\hat{\rho} + 2n_\nu v^\nu + (n_\nu v^\nu)^2 + 2n_\beta X^\beta n_\nu \partial^\nu U\right] \\ & + n_\nu X^\nu \partial_\beta(\hat{\rho}v^\beta) + \frac{1}{2}(n_\nu X^\nu)^2\,\partial_\beta\partial_\alpha(\hat{\rho}v^\alpha v^\beta) \\ & - \frac{1}{2}(n_\nu X^\nu)^2\,\partial_\beta[-\hat{\rho}\partial^\beta U + \partial^\beta p] + 2n_\nu n_\beta X^\beta \partial_\alpha(\hat{\rho}v^\alpha v^\nu) \\ & - 2n_\beta X^\beta n_\nu \partial^\nu p + \hat{\rho}O(\varepsilon^3). \end{aligned} \tag{21.6}$$

Substituting this into (21.3), integrating the result, and allowing for the fact that for static spherically symmetric objects

$$P_{(i)} = \frac{1}{3}\Omega_{(i)}, \quad \Omega^{\alpha\beta}_{(i)} = \frac{1}{3}\gamma^{\alpha\beta}\Omega_{(i)},$$

in relative variables (see Denisov and Logunov, 1982d) we obtain

$$\begin{aligned} E^{\alpha\beta} = {} & m\left[1 + \frac{4}{3}\,\frac{M_2\Omega_1 + M_1\Omega_2}{M}\right]R^\alpha R^\beta \\ & \times\left\{1 + \left(1 - \frac{3m}{M}\right)\frac{v^2}{2} + \frac{M_2 - M_1}{M}n_\nu v^\nu + \left(1 - \frac{2m}{M}\right)\frac{M}{R}\right\} \\ & + m\left(1 - \frac{3m}{M}\right)v^\alpha v^\beta (n_\nu R^\nu)^2 \\ & + m\,\frac{M_1 - M_2}{M}n_\nu R^\nu (v^\alpha R^\beta + v^\beta R^\alpha) + mR^2 O(\varepsilon^3), \end{aligned} \tag{21.7}$$

where $M = M_1 + M_2$ and $m = M_1M_2/M$. Using the Newtonian integral of energy

$$\frac{v^2}{2} - \frac{M}{R} = E = \text{const} + O(\varepsilon^4) \tag{21.8}$$

and introducing the notation

$$\widetilde{m} = m\left[1 + \left(1 - \frac{3m}{M}\right)E + \frac{4}{3}\,\frac{M_2\Omega_1 + M_1\Omega_2}{M}\right], \tag{21.9}$$

we can conveniently write (21.7) in the form

$$E^{\alpha\beta} = \widetilde{m} R^\alpha R^\beta \left\{1 + \left(2 - \frac{5m}{M}\right)\frac{m}{R} + \frac{M_2 - M_1}{M} n_\nu v^\nu\right\}$$
$$+ m \frac{M_1 - M_2}{M} n_\nu R^\nu (v^\alpha R^\beta + v^\beta R^\alpha)$$
$$+ m\left(1 - \frac{3m}{M}\right) v^\alpha v^\beta (n_\nu R^\nu)^2 + mR^2 O(\varepsilon^3). \quad (21.10)$$

To establish the components of the gravitational field, we find the second time derivative of (21.10) and, allowing for the post-Newtonian equations of motion, substitute this derivative into (21.2). As a result we obtain

$$\Phi^{\alpha\beta} = -\frac{2\widetilde{m}}{r}\left[P^\alpha_\nu P^\beta_\varepsilon - \frac{1}{2} P^{\alpha\beta} P_{\nu\varepsilon}\right]$$
$$\times \left\{2v^\nu v^\varepsilon \left[1 + \left(2 - \frac{5m}{M}\right)\frac{M}{R} - 2\left(1 - \frac{3m}{M}\right)\frac{M}{R^3}(n_\mu R^\mu)^2\right.\right.$$
$$\left. + \left(1 - \frac{3m}{M}\right)(n_\mu v^\mu)^2 + \frac{M_1 - M_2}{M} n_\mu v^\mu\right]$$
$$+ \frac{M}{R^3}(R^\nu v^\varepsilon + R^\varepsilon v^\nu)\left[\left(3 - \frac{10m}{M}\right) R_\mu v^\mu - 4\left(1 - \frac{3m}{M}\right) R_\mu v^\mu n_\delta v^\delta\right.$$
$$\left. - \frac{3(M_1 - M_2)}{M} n_\mu R^\mu - 3\left(1 - \frac{3m}{M}\right)(n_\mu R^\mu)^2 \frac{R_\delta v^\delta}{R^2}\right]$$
$$+ \frac{MR^\nu R^\varepsilon}{R^3}\left[-2 + 3\left(2 + \frac{3m}{M}\right)\frac{M}{R} - \left(4 + \frac{m}{M}\right) v^2 + 6\left(1 - \frac{3m}{M}\right)\frac{(R_\mu v^\mu)^2}{R^2}\right.$$
$$\left.\left. + 2\left(1 - \frac{3m}{M}\right)\frac{M(n_\mu R^\mu)^2}{R^3} - \frac{3(M_1 - M_2)}{M}\frac{R_\mu v^\mu n_\delta R^\delta}{R^2}\right]\right\} + O(\varepsilon^5). \quad (21.11)$$

To determine the coefficients k_1, k_2, and k_d in RTG we must retain in (21.11) only terms of the order of $m\varepsilon^2/r$:

$$\Phi^{\alpha\beta} = -\frac{4m}{r}\left[P^\alpha_\tau P^\beta_\sigma - \frac{1}{2} P^{\alpha\beta} P_{\tau\sigma}\right]\left\{v^\tau v^\sigma - \frac{MR^\tau R^\sigma}{R^3} + O(\varepsilon^3)\right\}.$$

Finding the second time derivative of this expression and substituting the derivative into (15.53), we arrive at the following expression for the intensity of energy emission in the form of gravitational waves by a compact binary system in RTG:

$$\frac{dI}{d\Omega} = \frac{m^2 M^2}{\pi R^6}\left\{4v^2 R^2 - 4R^2 (n_\nu v^\nu)^2 - 4v^2 (n_\nu R^\nu)^2 + 4(n_\nu v^\nu)^2 (n_\beta R^\beta)^2\right.$$
$$- \frac{15}{4}(R_\nu v^\nu)^2 - 6R_\nu v^\nu n_\beta v^\beta R_\alpha n^\alpha + \frac{6}{R^2} R_\nu v^\nu n_\beta v^\beta (R_\alpha n^\alpha)^3$$
$$\left. + \frac{3}{2R^2}(R_\nu v^\nu)^2 (R_\alpha n^\alpha)^2 + \frac{9}{4R^4}(R_\nu v^\nu)^2 (R_\alpha n^\alpha)^4\right\}.$$

Integrating with respect to the solid angle and allowing for

$$\int d\Omega n^\alpha n^\beta = -\frac{4\pi}{3}\gamma^{\alpha\beta},$$
$$\int d\Omega n^\alpha n^\beta n^\nu n^\delta = \frac{4\pi}{15}\left[\gamma^{\alpha\beta}\gamma^{\nu\delta} + \gamma^{\alpha\nu}\gamma^{\beta\delta} + \gamma^{\alpha\delta}\gamma^{\beta\nu}\right], \quad (21.12)$$
$$\int d\Omega n^\alpha n^\beta n^\nu = 0,$$

we get the following expression for the energy lost (per unit time) in a compact binary system to the emission of gravitational waves:

$$-\frac{dE}{dt}=\frac{8}{15}\frac{m^2M^2}{R^6}\left[12v^2R^2-11\left(R_\nu v^\nu\right)^2\right]. \tag{21.13}$$

Comparing (21.1) with (21.13), we obtain $k_1 = 12$, $k_2 = 11$, and $k_d = 0$. Thus, there can be no dipole gravitational waves in RTG, and the Peters-Mathews coefficients can be found from the results of observation of the double pulsar system PSR 1913+16.

A final remark is in order. If in GR we assume that the space-time coordinates are Cartesian (which is impossible in principle because no global Cartesian coordinates can be introduced in a Riemannian geometry), for the case of a weak gravitational field we obtain formula (21.13). This is explained by the fact that if we assume the space-time variables in GR to be Cartesian, the energy-momentum pseudotensor is equal to the gravitational-field energy-momentum tensor in RTG in Galilean coordinates. However, in other generally admissible coordinate systems this is not so, with the result that formula (21.13) for the energy loss does not, generally speaking, follow from GR. And, in general, what formula for radiation can there be in GR if selection of the proper reference frame is able to destroy the gravitational field? What we are dealing with here is one of the most profound and basic delusions in theoretical physics. Apparently, this is because dogmatism and faith penetrated GR so deeply and took such firm root that Einstein's ideas received no critical analysis or necessary creative theoretical development for a very long time. It will take considerable time before this dogmatism passes into history.

Appendix 1

Here we demonstrate that the Riemann-Christoffel curvature tensor R^k_{mnp}, the Ricci tensor R_{mn}, and the scalar curvature R are form-invariant under the $\partial_n \to D_n$ transformation, where D_n is the operator of covariant differentiation with respect to the metric of the Minkowski space-time. We also express R_{mn} and R in terms of $\tilde{g}^{mn}$ and give other useful expressions used in the main text.

The definition of the Riemann-Christoffel curvature tensor is

$$R^k_{mnp} = \partial_p \Gamma^k_{mn} - \partial_n \Gamma^k_{mp} + \Gamma^l_{mn}\Gamma^k_{lp} - \Gamma^l_{mp}\Gamma^k_{ln}, \tag{A1.1}$$

where the connection coefficients Γ^k_{mn} are expressed in terms of the metric tensor g_{mn} in the following manner:

$$\Gamma^k_{mn} = \frac{1}{2} g^{kq} (\partial_m g_{qn} + \partial_n g_{qm} - \partial_q g_{mn}). \tag{A1.2}$$

According to RTG, the variables x^n on which g_{mn}, g^{mn}, ∂_n and other quantities depend are curvilinear coordinates in the Minkowski space-time. Hence, combining

$$\partial_n g_{pk} = D_n g_{pk} + \gamma^q_{np} g_{qk} + \gamma^q_{nk} g_{qp} \tag{A1.3}$$

with (A1.2) and performing relatively simple calculations yields

$$\Gamma^k_{mn} = G^k_{mn} + \gamma^k_{mn}, \tag{A1.4}$$

where the connection coefficients γ^k_{mn} can be expressed in terms of the metric tensor of the Minkowski space-time, γ_{mn}, thus:

$$\gamma^k_{mn} = \frac{1}{2} \gamma^{kq} (\partial_m \gamma_{qn} + \partial_n \gamma_{qm} - \partial_q \gamma_{mn}), \tag{A1.5}$$

and

$$G^k_{mn} = \frac{1}{2} g^{kq} (D_m g_{qn} + D_n g_{qm} - D_q g_{mn}). \tag{A1.6}$$

Note that the G^k_{mn} constitute a third-rank tensor with respect to general coordinate transformations.

If (A1.4) is substituted into (A1.1), we get

$$R^k_{mnp} = \partial_p G^k_{mn} - \partial_n G^k_{mp} + G^l_{mn} G^k_{lp} - G^l_{mp} G^k_{ln} + \gamma^k_{lp} G^l_{mn} + \gamma^l_{mn} G^k_{lp} \\ - \gamma^k_{ln} G^l_{mp} - \gamma^l_{mp} G^k_{ln} + \partial_p \gamma^k_{mn} - \partial_n \gamma^k_{mp} + \gamma^l_{mn} \gamma^k_{lp} - \gamma^l_{mp} \gamma^k_{ln}.$$

The last four terms in this expression constitute the curvature tensor of flat space and, hence, are identically zero, so that we get

$$R^k_{mnp} = \partial_p G^k_{mn} - \partial_n G^k_{mp} + G^l_{mn} G^k_{lp} - G^l_{mp} G^k_{ln} + \gamma^k_{lp} G^l_{mn} + \gamma^l_{mn} G^k_{lp} \\ - \gamma^k_{ln} G^l_{mp} - \gamma^l_{mp} G^k_{ln}. \tag{A1.7}$$

As noted earlier, the G^k_{mn} constitute a tensor, in view of which for $\partial_p G^k_{mn}$ we have

$$\partial_p G^k_{mn} = D_p G^k_{mn} - \gamma^k_{pl} G^l_{mn} + \gamma^l_{pm} G^k_{ln} + \gamma^l_{pn} G^k_{lm}. \qquad (A1.8)$$

Substitution of (A1.8) into (A1.7) yields

$$R^k_{mnp} = D_p G^k_{mn} - D_n G^k_{mp} + G^l_{mn} G^k_{lp} - G^l_{mp} G^k_{ln}. \qquad (A1.9)$$

We see that the Riemann-Christoffel curvature tensor does not change under the $\partial_p \to D_p$ transformation or, in other words, R^k_{mnp} is form-invariant under such transformation.

Contracting (A1.9) on the indices k and p yields the Ricci tensor

$$R_{mn} = D_p G^p_{mn} - D_n G^p_{mp} + G^l_{mn} G^p_{lp} - G^l_{mp} G^p_{ln}. \qquad (A1.10)$$

Similarly, contraction of the Ricci tensor with the metric tensor g^{mn} yields the following expression for the scalar curvature R:

$$R = g^{mn} R_{mn} = g^{mn}(D_p G^p_{mn} - D_n G^p_{mp}) + g^{mn}(G^l_{mn} G^p_{lp} - G^l_{mp} G^p_{ln}). \qquad (A1.11)$$

Comparing (A1.10) and (A1.11) with the standard expressions for the Ricci tensor,

$$R_{mn} = \partial_p \Gamma^p_{mn} - \partial_n \Gamma^p_{mp} + \Gamma^l_{mn} \Gamma^p_{lp} - \Gamma^l_{mp} \Gamma^p_{ln},$$

and the scalar curvature,

$$R = g^{mn} R_{mn} = g^{mn}(\partial_p \Gamma^p_{mn} - \partial_n \Gamma^p_{mp}) + g^{mn}(\Gamma^l_{mn} \Gamma^p_{lp} - \Gamma^l_{mp} \Gamma^p_{ln}),$$

we see that these quantities are form-invariant under the $\partial_n \to D_n$ transformation.

Let us now establish an important relationship between the covariant derivatives $D_n A^{kq\cdots}_{pm\cdots}$ and $\nabla_n A^{kq\cdots}_{pm\cdots}$, where ∇_n stands for the operator of covariant differentiation with respect to metric g_{mn}. By definition,

$$\begin{aligned}\nabla_n A^{kq\cdots}_{pm\cdots} = {} & \partial_n A^{kq\cdots}_{pm\cdots} + \Gamma^k_{nl} A^{lq\cdots}_{pm\cdots} + \Gamma^q_{nl} A^{kl\cdots}_{pm\cdots} + \cdots \\ & - \Gamma^l_{np} A^{kq\cdots}_{lm\cdots} - \Gamma^l_{nm} A^{kq\cdots}_{pl\cdots} - \cdots .\end{aligned}$$

Replacing the Γ^k_{mn} with (A1.4), we get

$$\begin{aligned}\nabla_n A^{kq\cdots}_{pm\cdots} = {} & \partial_n A^{kq\cdots}_{pm\cdots} + \gamma^k_{nl} A^{lq\cdots}_{pm\cdots} + \gamma^q_{nl} A^{kl\cdots}_{pm\cdots} + \cdots \\ & - \gamma^l_{np} A^{kq\cdots}_{lm\cdots} - \gamma^l_{nm} A^{kq\cdots}_{pl\cdots} - \cdots \\ & + G^k_{nl} A^{lq\cdots}_{pm\cdots} + G^q_{nl} A^{kl\cdots}_{pm\cdots} + \cdots \\ & - G^l_{np} A^{kq\cdots}_{lm\cdots} - G^l_{nm} A^{kq\cdots}_{pl\cdots} - \cdots .\end{aligned}$$

Combining this with the definition of a covariant derivative $D_n A^{kq\cdots}_{pm\cdots}$, we get

$$\begin{aligned}\nabla_n A^{kq\cdots}_{pm\cdots} = {} & D_n A^{kq\cdots}_{pm\cdots} + G^k_{nl} A^{lq\cdots}_{pm\cdots} + G^q_{nl} A^{kl\cdots}_{pm\cdots} + \cdots \\ & - G^l_{np} A^{kq\cdots}_{lm\cdots} - G^l_{nm} A^{kq\cdots}_{pl\cdots} - \cdots .\end{aligned} \qquad (A1.12)$$

If we recall that $\nabla_n g^{kq} \equiv 0$, then (A1.12) yields the following well-known formula:

$$D_n g^{kq} + G^k_{nl} g^{lq} + G^q_{nl} g^{kl} \equiv 0. \qquad (A1.12a)$$

Let us now express (A1.10) and (A1.11) in terms of the tensor densities

$$\tilde{g}^{nm} = \sqrt{-g}\, g^{mn}, \quad \tilde{g}_{mn} = \frac{1}{\sqrt{-g}}\, g_{mn}. \tag{A1.13}$$

First, however, we will derive a number of expressions for the densities of the metric tensors of the effective Riemann space-time, or (A1.13). Obviously,

$$\tilde{g}^{mn}\tilde{g}_{nk} = \delta^m_k. \tag{A1.14}$$

Since

$$D_p\delta^m_k = \partial_p\delta^m_k + \gamma^m_{pl}\delta^l_k - \gamma^l_{pk}\delta^m_l = 0, \tag{A1.15a}$$

(A1.14) yields

$$\begin{aligned} D_p(\tilde{g}^{mn}\tilde{g}_{nk}) &= \partial_p(\tilde{g}^{mn}\tilde{g}_{nk}) + \gamma^m_{pl}\tilde{g}^{ln}\tilde{g}_{nk} - \gamma^l_{pk}\tilde{g}^{mn}\tilde{g}_{nl} \\ &= \partial_p\tilde{g}^{mn}\tilde{g}_{nk} + \tilde{g}^{mn}\partial_p\tilde{g}_{nk} = 0. \end{aligned} \tag{A1.15b}$$

Consider the formula $\partial_p\tilde{g}^{mn} = \partial_p(\sqrt{-g}\, g^{mn})$. We write it in the form

$$\begin{aligned} \partial_p\tilde{g}^{mn} &= \partial_p\left[\sqrt{-\gamma}\left(\frac{\sqrt{-g}}{\sqrt{-\gamma}}\, g^{mn}\right)\right] \\ &= (\partial_p\sqrt{-\gamma})(\sqrt{g/\gamma}\, g^{mn}) + \sqrt{-\gamma}\,\partial_p(\sqrt{g/\gamma}\, g^{mn}). \end{aligned} \tag{A1.16}$$

But $\partial_p\sqrt{-\gamma} = \sqrt{-\gamma}\,\gamma^k_{pk}$ and, in view of the fact that $\sqrt{-g/(-\gamma)}\, g^{mn}$ is a tensor,

$$\partial_p\left(\frac{\sqrt{-g}}{\sqrt{-\gamma}}\, g^{mn}\right) = D_p\left(\frac{\sqrt{-g}}{\sqrt{-\gamma}}\, g^{mn}\right) - \frac{\sqrt{-g}}{\sqrt{-\gamma}}\,\gamma^m_{pk}g^{kn} - \frac{\sqrt{-g}}{\sqrt{-\gamma}}\,\gamma^n_{pk}g^{mk},$$

(A1.16) yields

$$\partial_p\tilde{g}^{mn} = D_p\tilde{g}^{mn} - \gamma^m_{pk}\tilde{g}^{kn} - \gamma^n_{pk}\tilde{g}^{mk} + \gamma^k_{pk}\tilde{g}^{mn}. \tag{A1.17}$$

Reasoning along similar lines, we can show that

$$\partial_p\tilde{g}_{mn} = D_p\tilde{g}_{mn} + \gamma^k_{pm}\tilde{g}_{kn} + \gamma^k_{pn}\tilde{g}_{km} - \gamma^k_{pk}\tilde{g}_{mn}. \tag{A1.18}$$

Substituting (A1.17) and (A1.18) into (A1.15b) yields

$$D_p(\tilde{g}^{mn}\tilde{g}_{nk}) = D_p\tilde{g}^{mn}\tilde{g}_{nk} + \tilde{g}^{mn}\tilde{D}_p\tilde{g}_{nk} = 0. \tag{A1.19}$$

We write (A1.17) in the form

$$D_p\tilde{g}^{mn} = \partial_p\sqrt{-g}\, g^{mn} + \sqrt{-g}\,\partial_p g^{mn} + \sqrt{-g}(\gamma^m_{pk}g^{kn} + \gamma^n_{pk}g^{mk} - \gamma^k_{pk}g^{mn}). \tag{A1.17a}$$

Combining $\partial_p\sqrt{-g} = \sqrt{-g}\,\Gamma^k_{pk}$ with (A1.4) and (A1.17a) yields

$$\begin{aligned} D_p\tilde{g}^{mn} = \sqrt{-g}\, G^k_{pk}g^{mn} &+ \sqrt{-g}\,(\partial_p g^{mn} + g^{kn}\Gamma^m_{pk} + g^{mk}\Gamma^n_{pk}) \\ &- \sqrt{-g}\,(g^{kn}G^m_{pk} + g^{mk}G^n_{pk}). \end{aligned} \tag{A1.17b}$$

On the basis of (A1.2) and (A1.6) combined with (A1.15b) and (A1.19) we can easily demonstrate that

$$\partial_p g^{mn} + g^{kn}\Gamma^m_{pk} + g^{mk}\Gamma^n_{pk} \equiv 0, \tag{A1.20}$$

and

$$g^{kn}G^m_{pk} + g^{mk}G^n_{pk} \equiv -D_p g^{mn}. \tag{A1.21}$$

Hence, (A1.17b) yields

$$D_p\tilde{g}^{mn}=\sqrt{-g}\,(D_pg^{mn}+G^k_{pk}g^{mn}). \tag{A1.22}$$

Similarly, starting with (A1.18), we can show that

$$D_p\tilde{g}_{mn}=\frac{1}{\sqrt{-g}}\,(D_pg_{mn}-G^k_{pk}g_{mn}). \tag{A1.23}$$

Let us express G^k_{mn} in terms of $\tilde{g}_{mn}$. To this end we take D_pg_{mn} from (A1.23) and substitute it into (A1.6). The result is

$$G^k_{mn}=\tilde{G}^k_{mn}+\frac{1}{2}\,\delta^k_mG^l_{nl}+\frac{1}{2}\,\delta^k_nG^l_{ml}-\frac{1}{2}\,\tilde{g}_{mn}\tilde{g}^{kp}G^l_{pl}, \tag{A1.24}$$

where

$$\tilde{G}^k_{mn}=\frac{1}{2}\,\tilde{g}^{kp}\,(D_m\tilde{g}_{pn}+D_n\tilde{g}_{pm}-D_p\tilde{g}_{mn}). \tag{A1.25}$$

Contraction of (A1.24) on indices k and n yields

$$G^n_{mn}=-\tilde{G}^n_{mn}, \tag{A1.26}$$

which means that (A1.24) can be represented in the form

$$G^k_{mn}=\tilde{G}^k_{mn}-\frac{1}{2}\,\delta^k_m\tilde{G}^l_{nl}-\frac{1}{2}\,\delta^k_n\tilde{G}^l_{ml}+\frac{1}{2}\,\tilde{g}_{mn}\tilde{g}^{kp}\tilde{G}^l_{pl}, \tag{A1.27}$$

where

$$\tilde{G}^k_{mk}=\frac{1}{2}\,\tilde{g}^{kn}D_m\tilde{g}_{kn}. \tag{A1.28}$$

Substituting (A1.27) into (A1.10), we arrive at the following expression for the Ricci tensor:

$$\begin{aligned}R_{mn}&=D_p\tilde{G}^p_{mn}+\frac{1}{2}\,D_p\,(\tilde{g}_{mn}\tilde{g}^{pk}\tilde{G}^l_{kl})-\tilde{G}^l_{pm}\tilde{G}^p_{ln}+\frac{1}{2}\,\tilde{G}^q_{mq}\tilde{G}^p_{pn}\\&\quad-\frac{1}{2}\,\tilde{g}_{ln}\tilde{g}^{pk}\tilde{G}^l_{mp}\tilde{G}^q_{kq}-\frac{1}{2}\,\tilde{g}_{ml}\tilde{g}^{pk}\tilde{G}^l_{np}\tilde{G}^q_{kq}.\end{aligned} \tag{A1.29}$$

If we combine this with the RTG equation

$$D_p\tilde{g}^{pk}=0, \tag{A1.30}$$

we obtain

$$\begin{aligned}\overset{(0)}{R}_{mn}&=\frac{1}{2}\,\tilde{g}^{pk}D_p\,(D_m\tilde{g}_{kn}+D_n\tilde{g}_{km}-D_k\tilde{g}_{mn})+\frac{1}{2}\,\tilde{g}^{pk}D_p\,(\tilde{g}_{mn}\tilde{G}^l_{kl})\\&\quad-\tilde{G}^l_{mp}\tilde{G}^p_{ln}+\frac{1}{2}\,\tilde{G}^q_{mq}\tilde{G}^k_{nk}-\frac{1}{2}\,\tilde{g}_{ln}\tilde{g}^{pk}\tilde{G}^l_{mp}\tilde{G}^q_{kq}-\frac{1}{2}\,\tilde{g}_{ml}\tilde{g}^{pk}\tilde{G}^l_{np}\tilde{G}^q_{kq}.\end{aligned} \tag{A1.31}$$

Here and in what follows the symbol (0) above the Ricci tensor and above R signifies that Eq. (A1.30) has been taken into account.

Let us now go on from quantities with the tilde mark to quantities without that mark in (A1.31). Combining (A1.23), (A1.24), and (A1.26) with (A1.31), we get

$$\begin{aligned}\overset{(0)}{R}_{mn}&=\frac{1}{2}\,g^{pk}D_p\,(D_mg_{kn}+D_ng_{km}-D_kg_{mn}-G^q_{mq}g_{kn}-G^q_{nq}g_{km})\\&\quad+G^p_{np}G^q_{mq}-G^q_{mp}G^p_{nq}.\end{aligned} \tag{A1.32}$$

Contraction of (A1.32) with g^{mn} yields

$$\overset{(0)}{R} = g^{mn}\overset{(0)}{R}_{mn} = g^{pk}g^{mn}D_p\left(D_m g_{kn} - \frac{1}{2}D_k g_{mn} - G^q_{mq}g_{kn}\right) + g^{mn}(G^p_{np}G^q_{mq} - G^q_{mp}G^p_{nq}). \tag{A1.33}$$

If we use Eqs. (A1.3) and (A1.4) and perform certain transformations, we can write (A1.32) and (A1.33) as follows:

$$\overset{(0)}{R}_{mn} = \frac{1}{2}g^{pk}[\partial_p(\partial_m g_{kn} + \partial_n g_{km} - \partial_k g_{mn} - \Gamma^q_{mq}g_{kn} - \Gamma^q_{nq}g_{km})] + \Gamma^p_{np}\Gamma^q_{mq} - \Gamma^q_{mp}\Gamma^p_{nq}$$
$$- \left\{g^{pk}\left[g_{ql}\gamma^q_{pk}\Gamma^l_{mn} - \frac{1}{2}g_{nl}\gamma^l_{pk}\Gamma^q_{mq} - \frac{1}{2}g_{ml}\gamma^l_{pk}\Gamma^q_{nq}\right]\right\}, \tag{A1.34}$$

$$\overset{(0)}{R} = g^{pk}g^{mn}\left[\partial_p\left(\partial_m g_{kn} - \frac{1}{2}\partial_k g_{mn} - \Gamma^q_{mq}g_{kn}\right)\right] + g^{mn}(\Gamma^p_{np}\Gamma^q_{mq} - \Gamma^q_{mp}\Gamma^p_{nq})$$
$$- \{g^{pk}g^{mn}[g_{ql}\gamma^q_{pk}\Gamma^l_{mn} - g_{nl}\gamma^l_{pk}\Gamma^q_{mq}]\}. \tag{A1.35}$$

These formulas show that if in the expressions for the Ricci tensor and the scalar curvature R we allow for the RTG equation (A1.30), $\overset{(0)}{R}_{mn}$ and $\overset{(0)}{R}$ cease to be form-invariant under $D_m \leftrightarrow \partial_m$ transformations since the expressions inside the braces in (A1.34) and (A1.35) are not identically zero.

Thus, we see that allowing for Eq. (8.37) in Eqs. (8.36) makes the latter dependable on the metric tensor of the Minkowski space-time explicitly.

Various questions concerned with RTG require using different forms of the Ricci tensor R_{mn} and the scalar curvature R. A starting point for obtaining the different forms of R_{mn} and R is the Riemann-Cristoffel curvature tensor in form (A1.9). Since

$$D_pG^k_{mn} - D_nG^k_{mp} = \frac{1}{2}D_pg^{kq}(D_mg_{qn} + D_ng_{qm} - D_qg_{mn})$$
$$- \frac{1}{2}D_ng^{kq}(D_mg_{pq} + D_pg_{mq} - D_qg_{mp})$$
$$+ \frac{1}{2}g^{kq}(D_pD_mg_{qn} - D_pD_qg_{mn} - D_nD_mg_{pq} + D_nD_qg_{mp}), \tag{A1.36}$$

for $R_{lmnp} = g_{lk}R^k_{mnp}$ we have, in view of (A1.6), (A1.9), and (A1.36), the following:

$$R_{lmnp} = \frac{1}{2}(D_pD_mg_{ln} - D_pD_lg_{mn} - D_nD_mg_{pl} + D_nD_lg_{mp})$$
$$- D_pg_{lq}G^q_{mn} + D_ng_{lq}G^q_{mp} + g_{lk}(G^q_{mn}G^k_{qp} - G^q_{mp}G^k_{qn}), \tag{A1.37}$$

where we have allowed for the formula

$$g^{kq}D_pg_{ql} = -g_{ql}D_pg^{kq}, \tag{A1.38}$$

which can easily be obtained from the condition $g^{qk}g_{ql} = \delta^k_l$ combined with (A1.15a) and (A1.3).

Directly substituting (A1.6) we can show that

$$\begin{aligned} -D_pg_{lq}G^q_{mn} + g_{lk}G^q_{mn}G^k_{qp} &= -g_{kq}G^k_{mn}G^q_{pl}, \\ D_ng_{lq}G^q_{mp} - g_{lk}G^q_{mp}G^k_{qn} &= g_{kq}G^k_{mp}G^q_{nl}, \end{aligned} \tag{A1.39}$$

whence

$$R_{lmnp} = \frac{1}{2}(D_pD_mg_{ln} - D_pD_lg_{mn} - D_nD_mg_{pl} + D_nD_lg_{mp})$$
$$- g_{kq}(G^k_{mn}G^q_{pl} - G^k_{mp}G^q_{nl}). \qquad (A1.40)$$

Convolution of (A1.40) with tensor g^{lp} brings us to the following formula for the Ricci tensor:

$$R_{mn} = g^{lp}R_{lmnp} = \frac{1}{2} g^{lp}(D_pD_mg_{ln} - D_pD_lg_{mn} - D_nD_mg_{pl} + D_nD_lg_{mp})$$
$$- g^{lp}g_{kq}(G^k_{mn}G^q_{pl} - G^k_{mp}G^q_{nl}). \qquad (A1.41)$$

We introduce

$$G^q = g^{pl}G^q_{pl} = g^{pl}g^{qk}\left(D_pg_{kl} - \frac{1}{2}D_kg_{pl}\right), \qquad (A1.42)$$

which is a contravariant vector. If we take into account (A1.38), we can represent (A1.42) in the form

$$G^q = -(D_kg^{qk} + G^l_{kl}g^{qk}). \qquad (A1.43)$$

Comparing this with (A1.22), we get

$$G^q = -\frac{1}{\sqrt{-g}}D_k\tilde{g}^{kq}. \qquad (A1.44)$$

Note that if $\tilde{g}^{kq}$ satisfies the RTG equation (8.37), then $G^q = 0$.

Combining (A1.43) with (A1.38), we can easily arrive at the following expression for the covariant vector G_p:

$$G_p = g_{pq}G^q = g^{qk}D_kg_{pq} - G^l_{pl}. \qquad (A1.45)$$

Later we will need the equation

$$g^{pq}D_ng_{pq} = D_n \ln(-g). \qquad (A1.46)$$

Let us prove its validity. Since (see (A1.4) and (A1.6))

$$G^k_{nk} = \frac{1}{2}g^{lk}D_ng_{lk} = \Gamma^k_{nk} - \gamma^k_{nk}, \qquad (A1.47)$$

using the well-known representations $\Gamma^k_{nk} = \frac{1}{2}\partial_n \ln(-g)$ and $\gamma^k_{nk} = \frac{1}{2} \times \partial_n \ln(-\gamma)$, we obtain

$$\frac{1}{2}g^{lk}D_ng_{lk} = \frac{1}{2}\partial_n \ln\left(\frac{-g}{-\gamma}\right). \qquad (A1.48)$$

Since $\ln(-g/(-\gamma))$ is a scalar, ∂_n on the right-hand side of (A1.48) can be replaced with D_n, and since $D_n \ln(-\gamma) \equiv 0$, formula (A1.48) yields (A1.46). Thus, in view of (A1.48), we have the following representation:

$$G^k_{nk} = \frac{1}{2}D_n \ln(-g). \qquad (A1.49)$$

Note that $D_n \ln(-g)$ is a covariant vector.

Acting with D_m on (A1.46), we get

$$g^{pq}D_mD_ng_{pq} = D_mD_n \ln(-g) - D_mg^{pq}D_ng_{pq}. \qquad (A1.50)$$

Note that since D_m and D_n commute, (A1.50) implies

$$D_m g^{pq} D_n g_{pq} = D_n g^{pq} D_m g_{pq}. \tag{A1.51}$$

We could have arrived at the same result if (A1.38) was used as the starting equation, since the latter can be written in somewhat different forms:

$$D_p g_{nl} = -g_{kn} g_{ql} D_p g^{kq} \tag{A1.52a}$$

and

$$D_p g^{nl} = -g^{nk} g^{ql} D_p g_{kq}. \tag{A1.52b}$$

Substituting (A1.49) into (A1.45) and taking the covariant derivative with respect to x^l, we obtain

$$-g^{qk} D_l D_k g_{pq} = D_l g^{qk} D_k g_{pq} - D_l G_p - \frac{1}{2} D_l D_p \ln(-g). \tag{A1.53}$$

Since

$$D_l g^{qk} D_k g_{pq} \equiv \frac{1}{2} D_l g^{qk} (D_k g_{pq} + D_q g_{pk})$$
$$= D_l g^{qk} \left(g_{pr} G^r_{kq} + \frac{1}{2} D_p g_{kq} \right),$$

we can rewrite (A1.53) as

$$-g^{qk} D_l D_k g_{pq} = D_l g^{qk} g_{pr} G^r_{kq} + \frac{1}{2} D_l g^{qk} D_p g_{kq} - D_l G_p - \frac{1}{2} D_l D_p \ln(-g). \tag{A1.54}$$

Allowing for (A1.50) and (A1.54), we get

$$\frac{1}{2} g^{lp} (D_p D_m g_{ln} - D_n D_m g_{pl} + D_n D_l g_{mp})$$
$$= \frac{1}{2} (D_m G_n + D_n G_m) - \frac{1}{2} G^r_{lp} (D_m g^{lp} g_{nr} + D_n g^{lp} g_{mr}). \tag{A1.55}$$

Let us define the covariant second-rank tensor G_{mn} thus:

$$G_{mn} = \frac{1}{2} (D_m G_n + D_n G_m) - G^k_{mn} G_k. \tag{A1.56}$$

Then, combining (A1.55) and (A1.56) with (A1.41), we obtain for R_{mn} the following formula:

$$R_{mn} = -\frac{1}{2} g^{lp} D_p D_l g_{mn} - \frac{1}{2} G^r_{pl} (D_m g^{lp} g_{nr} + D_n g^{lp} g_{mr})$$
$$+ G_{mn} + g^{lp} g_{kq} G^k_{mp} G^q_{nl}. \tag{A1.57}$$

We see that on the right-hand side of (A1.57) only the first term contains a covariant second-order derivative of g_{mn} and that in terms of D_p this term has the structure of the d'Alembertian operator. The other terms contain only covariant first-order derivatives of the metric tensor of the Riemann space-time.

Let us now employ G^k_{mn} to introduce other tensor quantities:

$$G_{p,\,mn} = g_{pk} G^k_{mn} = \frac{1}{2} (D_m g_{pn} + D_n g_{pm} - D_p g_{mn}), \tag{A1.58}$$

$$G^{kq}_p = g^{km} g^{qn} G_{p,\,mn}, \tag{A1.59}$$

$$G^{l,\,kq} = g^{km} g^{qn} G^l_{mn} = g^{km} g^{qn} g^{lp} G_{p,\,mn}. \tag{A1.60}$$

Combining (A1.60) with (A1.52b), we can easily obtain the following representation for $G^{l,kq}$:

$$G^{l,kq}=\frac{1}{2}(g^{lp}D_pg^{kq}-g^{km}D_mg^{lq}-g^{qn}D_ng^{lk}). \quad \text{(A1.61)}$$

We define the covariant second-rank tensor A_{mn} thus:

$$A_{mn}=g^{pq}g^{lk}(D_pg_{ml}D_qg_{nk}-G_{l,mq}G_{k,np}). \quad \text{(A1.62)}$$

If we now allow for the formula

$$D_ng_{pm}=G_{p,mn}+G_{m,pn}, \quad \text{(A1.63)}$$

which can easily be derived from (A1.58), we find that

$$A_{mn}=g^{pq}g^{lk}(G_{l,mp}G_{n,qk}+G_{m,pl}D_qg_{nk}). \quad \text{(A1.64)}$$

If we employ (A1.59) and perform certain transformations, A_{mn} can be written in a somewhat different form:

$$A_{mn}=\frac{1}{2}G_n^{pl}D_mg_{lp}+\frac{1}{2}G_m^{qk}D_ng_{qk}+G_m^{qk}G_{n,qk}. \quad \text{(A1.65)}$$

On the basis of (A1.52b) and (A1.59) we can easily establish that

$$G_n^{pl}D_mg_{lp}=-D_mg^{lp}G_{n,pl}.$$

Then (A1.65) yields

$$A_{mn}=-\frac{1}{2}D_mg^{lp}G_{n,pl}-\frac{1}{2}D_ng^{qk}G_{m,qk}+G_m^{qk}G_{n,qk}. \quad \text{(A1.66)}$$

Equating (A1.62) with (A1.66), we get

$$g^{pq}g^{lk}D_pg_{ml}D_qg_{nk}-G_m^{kq}G_{n,kq}=-\frac{1}{2}D_mg^{lp}G_{n,pl}-\frac{1}{2}D_ng^{qk}G_{m,qk}+g^{pq}G_{mq}^kG_{k,np}. \quad \text{(A1.67)}$$

If we allow for this identity in (A1.57), we arrive at the following representation for R_{mn}:

$$R_{mn}=-\frac{1}{2}g^{lp}D_pD_lg_{mn}+G_{mn}+g^{pq}g^{lk}D_pg_{ml}D_qg_{nk}-G_m^{kq}G_{n,kq}. \quad \text{(A1.68)}$$

Finding the covariant derivative of (A1.38) and performing relatively simple transformations, we get

$$g_{mk}g_{qn}D_pD_lg^{kq}=-D_pD_lg_{mn}+g^{qk}(D_lg_{qn}D_pg_{mk}+D_lg_{mk}D_pg_{qn}), \quad \text{(A1.69)}$$

whereby for R_{mn} we have an alternative representation:

$$R_{mn}=\frac{1}{2}g_{mk}g_{qn}g^{lp}D_pD_lg^{kq}+G_{mn}-G_m^{kq}G_{n,kq}. \quad \text{(A1.70)}$$

Raising the indices in this formula yields the following expression for the contravariant Ricci tensor:

$$R^{mn}=\frac{1}{2}g^{lp}D_pD_lg^{mn}+G^{mn}-G^{m,kq}G_{kq}^n, \quad \text{(A1.71)}$$

where

$$G^{mn}=g^{mk}g^{nq}G_{qk}=\frac{1}{2}(g^{mk}D_kG^n+g^{nk}D_kG^m-D_kg^{mn}G^k). \quad \text{(A1.72)}$$

Now let us write the scalar curvature in different forms. By definition, $R = g_{mn}R^{mn}$, so that (A1.71) implies

$$R = \frac{1}{2} g^{lp} g_{mn} D_p D_l g^{mn} + G - G^{kq}_m G^m_{kq}, \tag{A1.73}$$

where G stands for the convolution

$$G = g_{mn} G^{mn} = D_n G^n + G^l_{nl} G^n. \tag{A1.74}$$

Taking into account (A1.44) and (A1.49), we get

$$G = -\frac{1}{\sqrt{-g}} D_m D_n \tilde{g}^{mn}. \tag{A1.75}$$

If we write (A1.46) in the form $-g_{pq} D_n g^{pq} = D_n \ln(-g)$ and act on it with operator D_n, we get

$$g_{pq} D_n D_m g^{pq} = -D_m D_n \ln(-g) - D_m g_{pq} D_n g^{pq}.$$

Substituting this into (A1.73), we arrive at the following expression for R:

$$R = -\frac{1}{2} g^{lp} D_l D_p \ln(-g) - \frac{1}{2} g^{lp} D_l g_{kq} D_p g^{kq} + G - G^{kq}_m G^m_{kq}. \tag{A1.76}$$

It can be proved that the following identity holds true:

$$-\frac{1}{2} g^{lp} D_l g_{kq} D_p g^{kq} - G^{kq}_m G^m_{kq} = -\frac{1}{2} D_m g^{kq} G^m_{kq},$$

which after substitution into (A1.76) yields

$$R = -\frac{1}{2} g^{ln} D_l D_n \ln(-g) + G - \frac{1}{2} D_m g^{kq} G^m_{kq}. \tag{A1.77}$$

Adding and subtracting $(1/2)\, G^n D_n \ln(-g)$ on the right-hand side of (A1.77) and introducing the notation

$$K = -\frac{1}{2} D_m g^{kq} G^m_{kq} - \frac{1}{2} G^p D_p \ln(-g), \tag{A1.78}$$

we represent (A1.77) in the form

$$R = -\Box (\ln \sqrt{-g}) + G + K, \tag{A1.79}$$

where $\Box$ is the generalized d'Alembertian operator,

$$\Box = g^{mn} D_m D_n - G^n D_n = \frac{1}{\sqrt{-g}} D_m \{\sqrt{-g}\, g^{mn} D_n\}. \tag{A1.80}$$

It is easy to show that (A1.78) can be written in the form

$$K = g^{mn} (G^q_{mp} G^p_{nq} - G^q_{mn} G^p_{qp}). \tag{A1.81}$$

Comparing this expression with the RTG Lagrangian (8.26), we find that

$$L_g = -\frac{1}{16\pi} \sqrt{-g}\, K. \tag{A1.82}$$

In conclusion we note that since $D_p \tilde{g}^{pq}$ vanishes in RTG, we have $G = 0$, $G^{mn} = 0$, and $G_{mn} = 0$, and for the Ricci tensor and scalar curvature we have

in this case the following representations:

$$\overset{(0)}{R}_{mn} = \frac{1}{2} g_{mk} g_{nq} g^{lp} D_p D_l g^{kq} - G^{kq}_m G_{n,kq}, \quad (A1.83)$$

$$\overset{(0)}{R}{}^{mn} = \frac{1}{2} g^{lp} D_p D_l g^{mn} - G^{m,kq} G^n_{kq}, \quad (A1.84)$$

$$\overset{(0)}{R} = -\Box (\ln \sqrt{-g}) + K. \quad (A1.85)$$

On the basis of (A1.84) and (A1.85) we can easily derive an explicit expression for the Hilbert tensor:

$$\overset{(0)}{R}{}^{mn} - \frac{1}{2} g^{mn} \overset{(0)}{R} = \frac{1}{2} g^{lp} D_p D_l g^{mn} - G^{m,kq} G^n_{kq} + \frac{1}{2} g^{mn} \Box (\ln \sqrt{-g}) - \frac{1}{2} g^{mn} K. \quad (A1.86)$$

Note that the tensors (A1.83)-(A1.86) depend essentially on the metric tensor of the Minkowski space-time. They assume an especially simple form if Galilean coordinates are used in the Minkowski space-time.

Appendix 2

The postulated system of equations (8.3) for the gravitational field is not a corollary of the principle of least action. Being universal, therefore, these equations must be taken as "additional" conditions in the principle of least action. In other words, the variation of action

$$J = \int L d^4x \quad (A2.1)$$

over the fields $\tilde{\Phi}^{mn}$ or, in view of (8.1), over the $\tilde{g}^{mn}$ must be carried out on a manifold defined by the system of equations (8.3), $D_m \tilde{g}^{mn} = 0$.

It is well known that such a variational problem is solved by Lagrange's method of multipliers. A standard approach here is to add a term $\eta_m D_n \tilde{g}^{mn}$ to the Lagrangian L in the integrand of (A2.1), where the η_m are Lagrange's multipliers, and to apply the principle of least action to the integral

$$J = \int (L + \eta_n D_m \tilde{g}^{mn})\, d^4x. \quad (A2.2)$$

Since variation of (A2.2) means that we must vary the η_m and the components of the tensor density $\tilde{g}^{mn}$ independently, we find that

$$\frac{\delta L}{\delta \tilde{g}^{mn}} - \frac{1}{2} (D_n \eta_m + D_m \eta_n) = 0, \quad (A2.3)$$

$$D_n \tilde{g}^{mn} = 0. \quad (A2.4)$$

Further analysis will be carried out using the example of the Lagrangian $L = L_g + L_M$, where L_g is given by (8.22) and L_M is the matter Lagrangian. If for $L + \eta_m D_n \tilde{g}^{mn}$ we calculate the total energy-momentum tensor t^{mn} using for-

mula (6.17), we get

$$t^{mn}=J^{mn}+D_k[(\tilde{g}^{kn}\gamma^{ml}+\tilde{g}^{km}\gamma^{nl}-\tilde{g}^{mn}\gamma^{kl})\eta_l]$$
$$+2\sqrt{-\gamma}\left(\gamma^{ml}\gamma^{nk}-\frac{1}{2}\gamma^{mn}\gamma^{lk}\right)\left[\frac{\delta L}{\delta\tilde{g}^{kl}}-\frac{1}{2}D_l\eta_k-\frac{1}{2}D_k\eta_l\right].$$

We see that if Eq. (A2.3) holds true, then

$$t^{mn}=J^{mn}+D_k\left[\eta_l\left(\tilde{g}^{kn}\gamma^{ml}+\tilde{g}^{km}\gamma^{nl}-\tilde{g}^{mn}\gamma^{kl}\right)\right]. \tag{A2.5}$$

In addition to Eqs. (A2.3) and (A2.4), the following matter equations must hold true: $D_m t^{mn}=0$. Combining (A2.5) with (A2.4), we find that

$$\tilde{g}^{km}D_mD_k\eta^n=0,$$

which implies that Lagrange's multipliers η^n can be taken equal to zero.

Appendix 3

Let us consider the gravitational-field Lagrangian density of general form, quadratic in the first-order derivatives $D_l\tilde{g}^{mn}$.

Relativistic invariance implies that the total Lagrangian density can be represented in the form

$$L_g=\sum_{i=1}^{5}a_iL_i+\sum_{j=1}^{6}L_{gj}, \tag{A3.1}$$

where

$$L_1=\tilde{g}_{kq}D_m\tilde{g}^{pq}D_p\tilde{g}^{km},\quad L_2=\tilde{g}^{lp}D_l\tilde{g}^{mn}D_p\tilde{g}_{mn},$$
$$L_3=\tilde{g}_{km}\tilde{g}_{nq}\tilde{g}^{lp}D_l\tilde{g}^{km}D_p\tilde{g}^{nq},\quad L_4=\tilde{g}_{km}D_p\tilde{g}^{kp}D_n\tilde{g}^{mn}, \tag{A3.2}$$
$$L_5=\tilde{g}_{mn}D_p\tilde{g}^{pk}D_k\tilde{g}^{mn};$$

$$L_{g1}=\tilde{\gamma}_{km}\tilde{g}_{nq}\tilde{g}^{lp}\left[b_1\Lambda_{lp}^{(kq)(mn)}+c_1\Lambda_{lp}^{(km)(qn)}\right],$$
$$L_{g2}=\tilde{g}_{km}\tilde{g}_{nq}\tilde{\gamma}^{lp}\left[b_2\Lambda_{lp}^{(kq)(mn)}+c_2\Lambda_{lp}^{(km)(qn)}\right],$$
$$L_{g3}=\tilde{\gamma}_{mk}\tilde{\gamma}_{nq}\tilde{g}^{lp}\left[b_3\Lambda_{lp}^{(kq)(mn)}+c_3\Lambda_{lp}^{(km)(qn)}\right],$$
$$L_{g4}=\tilde{\gamma}_{mk}\tilde{g}_{nq}\tilde{\gamma}^{lp}\left[b_4\Lambda_{lp}^{(kq)(mn)}+c_4\Lambda_{lp}^{(km)(qn)}\right], \tag{A3.3}$$
$$L_{g5}=\tilde{\gamma}_{mk}\tilde{\gamma}_{nq}\tilde{\gamma}^{lp}\left[b_5\Lambda_{lp}^{(kq)(mn)}+c_5\Lambda_{lp}^{(km)(qn)}\right],$$
$$L_{g6}=b_6\tilde{\gamma}_{mk}D_q\tilde{g}^{kp}D_n\tilde{g}^{mn}+c_6\tilde{\gamma}_{mn}D_p\tilde{g}^{pq}D_q\tilde{g}^{mn}.$$

Here the coefficients a_i, b_j, and c_j are arbitrary numbers and

$$\Lambda_{lp}^{(kq)(mn)}\equiv D_l\tilde{g}^{kq}D_p\tilde{g}^{mn}. \tag{A3.4}$$

Note that the most general form, quadratic in the derivatives $D_m\tilde{g}^{lk}$, of the gravitational-field Lagrangian density can be set up if we take convolutions by employing expressions of the type $\gamma_{pk}g^{pl}\gamma_{ln}$, $g_{kp}\gamma^{pl}g_{ln}$, and the like. This, however, would not lead us to anything essentially new. Hence, we restrict our discussion to the Lagrangians (A3.2) and (A3.3).

Under a gauge transformation (10.5) an infinitesimal variation in the Lagrangian densities (A3.2) assumes the form

$$\delta_\varepsilon L_i = D_k Q^k_{(i)} + \varepsilon^k_{(x)} [\alpha^{(i)}_k + \beta^{(i)}_k + \sigma^{(i)}_k], \quad i = 1, 2, \ldots, 5, \tag{A3.5}$$

where

$$\begin{aligned}
Q^k_{(1)} &= \tilde g_{pq} D_m \tilde g^{pk} \delta_\varepsilon \tilde g^{mq} + \tilde g_{mq} D_p \tilde g^{kq} \delta_\varepsilon \tilde g^{pm} + \varepsilon^k \tilde g^{mq} D_p \tilde g_{nq} D_m \tilde g^{np} \\
&\quad - 2\varepsilon^m \tilde g^{kq} D_p (\tilde g_{nq} D_m \tilde g^{np}) - 2\varepsilon^n (\tilde g_{nq} D_m \tilde g^{pk} D_p \tilde g^{nq} + \tilde g^{kp} D_m \tilde g_{nq} D_p \tilde g^{mq}) \\
&\quad - 2\varepsilon^n \tilde g^{pk} \tilde g_{nq} D_m D_p \tilde g^{qm}, \\
Q^k_{(2)} &= 2\tilde g^{lk} D_l \tilde g_{mn} \delta_\varepsilon \tilde g^{mn} - 2\varepsilon^n \tilde g^{mk} D_p (\tilde g^{lp} D_l \tilde g_{mn}) + 2\varepsilon^q \tilde g_{nq} D_l (\tilde g^{lp} D_p \tilde g^{kn}) \\
&\quad + 2\varepsilon^p \tilde g^{lk} D_p \tilde g^{mn} D_l \tilde g_{mn} + \varepsilon^k \tilde g^{mn} D_p (\tilde g^{lp} D_l \tilde g_{mn}) \\
&\quad - \varepsilon^k \tilde g_{mn} D_p (\tilde g^{lp} D_l \tilde g^{mn}) - \varepsilon^k \tilde g^{lp} D_p \tilde g^{mn} D_l \tilde g_{mn}, \\
Q^k_{(3)} &= 8 D_m \varepsilon^m \tilde g^{kp} \tilde G^l_{pl} - 8\varepsilon^k D_l (\tilde g^{lp} \tilde G^m_{pm}) - 4\varepsilon^k \tilde g^{lp} \tilde G^m_{pm} \tilde G^n_{ln}.
\end{aligned} \tag{A3.6}$$

Since we have no need for the expressions for $Q^k_{(4)}$ and $Q^k_{(5)}$, they are not given here. Also

$$\alpha^{(1)}_k = 2\tilde g^{pl} \tilde g_{km} D_l D_n D_p \tilde g^{mn}, \quad \alpha^{(2)}_k = -2\alpha^{(1)}_k - \frac{1}{2}\alpha^{(3)}_k,$$

$$\alpha^{(3)}_k = -4\tilde g^{lp} \tilde g_{mn} D_k D_p D_l \tilde g^{mn}, \quad \alpha^{(4)}_k = \alpha^{(1)}_k, \quad \alpha^{(5)}_k = -\frac{1}{4}\alpha^{(3)}_k - 2 D_k D_l D_p \tilde g^{pl}. \tag{A3.7}$$

Of the various $\beta^{(i)}_k$ and $\sigma^{(i)}_k$ we will give only the expressions of those needed for our analysis:

$$\begin{aligned}
\beta^{(1)}_k &= 2\tilde g_{lk} D_q (D_m \tilde g^{pq} D_p \tilde g^{lm}) - 2\tilde g_{nq} D_l (D_p \tilde g^{nl} D_k \tilde g^{pq}) \\
&\quad - 2\tilde g^{lm} \tilde g_{kn} \tilde g_{qr} D_l (D_p \tilde g^{nr} D_m \tilde g^{pq}) + 2 D_l (\tilde g^{ml} \tilde g_{kq}) D_p D_m \tilde g^{pq} \\
&\quad + 2\tilde g_{nq} D_k \tilde g^{mn} D_p D_m \tilde g^{pq} + \tilde g_{nq} D_k (D_p \tilde g^{mn} D_m \tilde g^{pq}),
\end{aligned} \tag{A3.8}$$

$$\begin{aligned}
\beta^{(2)}_k &= 2 D_k (\tilde g_{mn} \tilde g^{lp}) D_l D_p \tilde g^{mn} - 4 D_n (\tilde g_{mk} \tilde g^{lp}) D_l D_p \tilde g^{mn} \\
&\quad + \tilde g^{mn} \tilde g^{lp} D_k (\tilde g_{rm} \tilde g_{nq}) D_p D_l \tilde g^{rq} - 4\tilde g_{mk} D_n (D_l \tilde g^{lp} D_p \tilde g^{mn}) \\
&\quad + 2\tilde g^{lp} \tilde g_{mq} \tilde g_{nr} D_p (D_k \tilde g^{mn} D_l \tilde g^{qr}) - 3\tilde g^{lp} \tilde g_{qm} \tilde g_{nr} D_k (D_p \tilde g^{mn} D_l \tilde g^{qr}) \\
&\quad + 4\tilde g^{lp} \tilde g_{mq} \tilde g_{kr} D_n (D_l \tilde g^{qr} D_p \tilde g^{mn}) + 2\tilde g_{mn} D_k (D_l \tilde g^{lp} D_p \tilde g^{mn}),
\end{aligned} \tag{A3.9}$$

$$\begin{aligned}
\beta^{(3)}_k &= -2\tilde g_{pm} \tilde g_{nq} \tilde g^{lr} D_r (D_l \tilde g^{pm} D_k \tilde g^{nq}) + 4 D_k (\tilde g_{mn} \tilde g^{lp}) D_l D_p \tilde g^{mn} \\
&\quad - 4\tilde g_{mn} D_k (D_p \tilde g^{lp} D_l \tilde g^{mn}) - 2\tilde g^{nq} D_k (\tilde g_{rm} \tilde g_{nq} \tilde g^{lp}) D_p D_l \tilde g^{mr} \\
&\quad + 4\tilde g_{qn} \tilde g_{mr} \tilde g^{lp} D_k (D_p \tilde g^{qr} D_l \tilde g^{mn}) + \tilde g_{mn} \tilde g_{qr} \tilde g^{lp} D_k (D_p \tilde g^{qr} D_l \tilde g^{mn}),
\end{aligned} \tag{A3.10}$$

$$\begin{aligned}
\beta^{(4)}_k &= 2 D_m (\tilde g_{kl} \tilde g^{nm}) D_n D_q \tilde g^{lq} + 2\tilde g_{mk} D_n \tilde g^{mn} D_l D_p \tilde g^{lp} \\
&\quad - 2\tilde g^{nr} \tilde g_{km} \tilde g_{lp} D_q \tilde g^{lq} D_r D_n \tilde g^{mp};
\end{aligned} \tag{A3.11}$$

$$\begin{aligned}
\sigma^{(1)}_k &= 2 D_q \tilde g_{nk} D_m \tilde g^{pq} D_p \tilde g^{mn} + 2 D_n \tilde g^{mn} D_p \tilde g_{kq} D_m \tilde g^{pq} \\
&\quad - 2\tilde g^{ml} \tilde g_{nk} D_l \tilde g_{rq} D_p \tilde g^{nr} D_m \tilde g^{pq} - 2\tilde g^{ml} \tilde g_{qr} D_l \tilde g_{nk} D_p \tilde g^{nr} D_m \tilde g^{pq},
\end{aligned} \tag{A3.12}$$

$$\begin{aligned}\sigma_k^{(2)} = &-4D_n\tilde{g}_{mk}D_l\tilde{g}^{lp}D_p\tilde{g}^{mn} - 4D_n\tilde{g}^{lp}D_l\tilde{g}_{mk}D_p\tilde{g}^{mn}\\ &+2D_k\tilde{g}_{mn}D_p\tilde{g}^{lp}D_l\tilde{g}^{mn} + 2D_k\tilde{g}^{lp}D_l\tilde{g}_{mn}D_p\tilde{g}^{mn}\\ &-4\tilde{g}^{lp}\tilde{g}_{mq}D_p\tilde{g}_{nr}D_k\tilde{g}^{mn}D_l\tilde{g}^{qr} + 4\tilde{g}^{lp}\tilde{g}_{mq}D_n\tilde{g}_{kr}D_l\tilde{g}^{qr}D_p\tilde{g}^{mn}\\ &+4\tilde{g}^{lp}\tilde{g}_{kr}D_n\tilde{g}_{mq}D_l\tilde{g}^{qr}D_p\tilde{g}^{mn},\end{aligned} \tag{A3.13}$$

$$\begin{aligned}\sigma_k^{(3)} = &-4D_k\tilde{g}_{mn}D_p\tilde{g}^{lp}D_l\tilde{g}^{mn} + 8\tilde{g}^{lp}\tilde{g}_{qn}D_k\tilde{g}_{mr}D_p\tilde{g}^{qr}D_l\tilde{g}^{mn}\\ &-4D_k\tilde{g}^{lp}D_p\tilde{g}_{mn}D_l\tilde{g}^{mn},\end{aligned} \tag{A3.14}$$

$$\sigma_k^{(4)} = 4D_m\tilde{g}^{mn}D_n\tilde{g}_{kl}D_q\tilde{g}^{lq} - 4\tilde{g}^{nr}\tilde{g}_{lp}D_r\tilde{g}_{km}D_n\tilde{g}^{mp}D_q\tilde{g}^{lq}. \tag{A3.15}$$

Similarly, under a gauge transformation (10.5) an infinitesimal variation of the Lagrangian densities (A3.3) assumes the form

$$\delta_\varepsilon L_{gj} = D_k\Theta^k_{(j)} + \varepsilon^k_{(x)}[b_j(U_k^{(j)} + V_k^{(j)} + W_k^{(j)}) + c_j(X_k^{(j)} + Y_k^{(j)} + Z_k^{(j)})], \quad j = 1, 2, \ldots, 6, \tag{A3.16}$$

where the $U_k^{(j)}$ and $X_k^{(j)}$ contain covariant third-order derivatives of $\tilde{g}^{mn}$ and are given by the following formulas:

$$\begin{aligned}
U_k^{(1)} &= 2\tilde{\gamma}_{mn}\tilde{g}^{lp}(\tilde{g}^{mr}\tilde{g}_{qk}D_rD_pD_l\tilde{g}^{nq} - D_lD_pD_k\tilde{g}^{mn}),\\
U_k^{(2)} &= 2\tilde{\gamma}^{lp}(2\tilde{g}_{mk}D_nD_pD_l\tilde{g}^{mn} - \tilde{g}_{mn}D_kD_lD_p\tilde{g}^{mn}),\\
U_k^{(3)} &= 2\tilde{\gamma}_{mn}\tilde{g}^{mr}\tilde{g}^{lp}(2\tilde{\gamma}_{kq}D_rD_pD_l\tilde{g}^{nq} - \tilde{\gamma}_{qr}D_kD_pD_l\tilde{g}^{nq}),\\
U_k^{(4)} &= 2\tilde{\gamma}^{lp}(\tilde{\gamma}_{mn}\tilde{g}^{mr}\tilde{g}_{kq}D_rD_pD_l\tilde{g}^{nq} + \tilde{\gamma}_{kn}D_mD_pD_l\tilde{g}^{mn} - \tilde{\gamma}_{mn}D_kD_pD_l\tilde{g}^{mn}),\\
U_k^{(5)} &= 2\tilde{\gamma}^{lp}\tilde{\gamma}_{mn}\tilde{g}^{mr}(2\tilde{\gamma}_{kq}D_rD_pD_l\tilde{g}^{nq} - \tilde{\gamma}_{rq}D_kD_pD_l\tilde{g}^{nq}),\\
U_k^{(6)} &= 2\tilde{\gamma}_{km}\tilde{g}^{lp}D_pD_qD_l\tilde{g}^{mq};\\
X_k^{(1)} &= \tilde{g}^{lp}(2\tilde{\gamma}_{mn}D_kD_pD_l\tilde{g}^{mn} - 2\tilde{\gamma}_{km}\tilde{g}^{mr}\tilde{g}_{nq}D_kD_lD_p\tilde{g}^{nq} + \tilde{\gamma}_{mr}\tilde{g}^{mr}\tilde{g}_{nq}D_kD_lD_p\tilde{g}^{nq}),\\
X_k^{(2)} &= -4\tilde{\gamma}^{lp}\tilde{g}_{mn}D_kD_pD_l\tilde{g}^{mn},\\
X_k^{(3)} &= 2\tilde{\gamma}_{rm}\tilde{g}^{nq}\tilde{g}^{lp}(2\tilde{\gamma}_{kn}D_qD_pD_l\tilde{g}^{mr} - \tilde{\gamma}_{nq}D_kD_pD_l\tilde{g}^{mr}),\\
X_k^{(4)} &= -\tilde{\gamma}^{lp}(2\tilde{\gamma}_{mn}D_kD_pD_l\tilde{g}^{mn} - 2\tilde{\gamma}_{mk}\tilde{g}^{mr}\tilde{g}_{nq}D_rD_lD_p\tilde{g}^{nq} + \tilde{\gamma}_{mr}\tilde{g}^{mr}\tilde{g}_{nq}D_kD_lD_p\tilde{g}^{nq}),\\
X_k^{(5)} &= 2\tilde{\gamma}_{nq}\tilde{\gamma}^{lp}\tilde{g}^{mr}(2\tilde{\gamma}_{km}D_rD_lD_p\tilde{g}^{nq} - \tilde{\gamma}_{mr}D_kD_lD_p\tilde{g}^{nq}),\\
X_k^{(6)} &= \tilde{g}^{mn}(2\tilde{\gamma}_{mk}D_nD_pD_q\tilde{g}^{pq} - \tilde{\gamma}_{mn}D_kD_pD_q\tilde{g}^{pq} + \tilde{\gamma}_{pq}D_kD_mD_n\tilde{g}^{pq}).
\end{aligned} \tag{A3.17}$$

Since we have no need for the expressions for the $V_k^{(j)}$, $W_k^{(j)}$, $Y_k^{(j)}$, $Z_k^{(j)}$, and $\Theta^k_{(j)}$, $j = 1, 2, \ldots, 6$, they are not given here. The only property of these quantities that will be used in our discussion is the fact that the $V_k^{(j)}$ and $Y_k^{(j)}$ contain in each term both the first- and second-order derivatives of $\tilde{g}^{mn}$, while the $W_k^{(j)}$ and $Z_k^{(j)}$ contain only first-order derivatives.

Combining (A3.5) and (A3.16) with (A3.1) yields

$$\delta_\varepsilon L_g = D_k \Big[\sum_{i=1}^{5} a_i Q^k_{(i)} + \sum_{j=1}^{6} \Theta^k_{(j)} \Big] + \varepsilon^k(x) \Big\{ \sum_{i=1}^{5} a_i (\alpha_k^{(i)} + \beta_k^{(i)} + \sigma_k^{(i)}) + \sum_{j=1}^{6} [b_j (U_k^{(j)} + V_k^{(j)} + W_k^{(j)}) + c_j (X_k^{(j)} + Y_k^{(j)} + Z_k^{(j)})] \Big\}. \quad \text{(A3.18)}$$

We see that L_g satisfies the gauge principle if and only if

$$\sum_{i=1}^{5} a_i (\alpha_k^{(i)} + \beta_k^{(i)} + \sigma_k^{(i)}) + \sum_{j=1}^{6} [b_j (U_k^{(j)} + V_k^{(j)} + W_k^{(j)}) + c_j (X_k^{(j)} + Y_k^{(j)} + Z_k^{(j)})] \equiv 0. \quad \text{(A3.19)}$$

Since (a) the $\alpha_k^{(i)}$, $U_k^{(j)}$, and $X_k^{(j)}$ contain third-order derivatives of g^{mn}, (b) the $\beta_k^{(i)}$, $V_k^{(j)}$, and $Y_k^{(j)}$ contain second-order derivatives, and, finally, (c) the $\sigma_k^{(i)}$, $W_k^{(j)}$, and $Z_k^{(j)}$ contain only first-order derivatives, condition (A3.19) splits into separate identities:

$$\sum_{i=1}^{5} a_i \alpha_k^{(i)} + \sum_{j=1}^{6} (b_j U_k^{(j)} + c_j X_k^{(j)}) \equiv 0, \quad \text{(A3.20)}$$

$$\sum_{i=1}^{5} a_i \beta_k^{(i)} + \sum_{j=1}^{6} (b_j V_k^{(j)} + c_j Y_k^{(j)}) \equiv 0, \quad \text{(A3.21)}$$

$$\sum_{i=1}^{5} a_i \sigma_k^{(i)} + \sum_{j=1}^{6} (b_j W_k^{(j)} + c_j Z_k^{(j)}) \equiv 0. \quad \text{(A3.22)}$$

Both $U_k^{(j)}$ and $X_k^{(j)}$ contain the metric tensor of the Minkowski space-time, but $\alpha_k^{(i)}$ does not, in view of which (A3.20) splits into two independent identities:

$$\sum_{i=1}^{5} a_i \alpha_k^{(i)} \equiv 0, \quad \text{(A3.23)}$$

$$\sum_{j=1}^{6} (b_j U_k^{(j)} + c_j X_k^{(j)}) \equiv 0. \quad \text{(A3.24)}$$

Combining (A3.7) with (A3.23) yields

$$\alpha_k^{(1)} (a_1 - 2a_2 + a_4) + \alpha_k^{(3)} \left(a_3 - \frac{1}{2} a_2 \right) + \alpha_k^{(5)} a_5 \equiv 0.$$

Since $\alpha_k^{(1)}$, $\alpha_k^{(3)}$, and $\alpha_k^{(5)}$ are independent of each other, we have

$$a_1 - 2a_2 + a_4 = 0, \quad a_3 - \frac{1}{2} a_2 = 0, \quad a_5 = 0. \quad \text{(A3.25)}$$

Equations (A3.17) demonstrate that the $U^{(j)}_k$ and $X^{(j)}_k$ are independent of each other, whereby (A3.24) implies

$$b_j = c_j = 0, \quad j = 1, 2, \ldots, 6. \quad \text{(A3.26)}$$

Allowing for (A3.25) and (A3.26) in identities (A3.21) and (A3.22), we get

$$a_1\left(\beta_k^{(1)}+\frac{1}{2}\beta_k^{(2)}+\frac{1}{4}\beta_k^{(3)}\right)+a_4\left(\frac{1}{2}\beta_k^{(2)}+\frac{1}{4}\beta_k^{(3)}+\beta_k^{(4)}\right)\equiv 0, \tag{A3.27}$$

$$a_1\left(\sigma_k^{(1)}+\frac{1}{2}\sigma_k^{(2)}+\frac{1}{4}\sigma_k^{(3)}\right)+a_4\left(\frac{1}{2}\sigma_k^{(2)}+\frac{1}{4}\sigma_k^{(3)}+\sigma_k^{(4)}\right)\equiv 0. \tag{A3.28}$$

On the basis of (A3.8)-(A3.14) one can establish (through somewhat laborious manipulations) that

$$\beta_k^{(1)}+\frac{1}{2}\beta_k^{(2)}+\frac{1}{4}\beta_k^{(3)}\equiv 0, \tag{A3.29}$$

$$\sigma_k^{(1)}+\frac{1}{2}\sigma_k^{(2)}+\frac{1}{4}\sigma_k^{(3)}\equiv 0. \tag{A3.30}$$

Hence, as (A3.27) and (A3.28) clearly show, the coefficient a_1 is not determined by the two. Its value is determined by the correspondence principle.

If we substitute (A3.30) into (A3.28), we get

$$a_4\left(\sigma_k^{(4)}-\sigma_k^{(1)}\right)\equiv 0. \tag{A3.31}$$

Since the difference $\sigma_k^{(4)}-\sigma_k^{(1)}$ is not identically zero (see (A3.12) and (A3.15)), we conclude that

$$a_4=0. \tag{A3.32}$$

The final formulas for the coefficients a_i ($i=1, 2, \ldots, 5$) are

$$a_2=\frac{1}{2}a_1, \quad a_3=\frac{1}{4}a_1, \quad a_4=a_5=0. \tag{A3.33}$$

Thus, if the gauge principle formulated in Chapter 10 is taken as the starting point of our discussion, then, in view of (A3.26) and (A3.33), the general form (A3.1) of the Lagrangian density leads unambiguously to the following expression for the Lagrangian density:

$$L_g=a_1\left[\tilde{g}_{kq}D_m\tilde{g}^{pq}D_p\tilde{g}^{km}+\frac{1}{2}\tilde{g}^{lp}D_l\tilde{g}_{mn}D_p\tilde{g}^{mn}+\frac{1}{4}\tilde{g}_{km}\tilde{g}_{nq}\tilde{g}^{lp}D_l\tilde{g}^{km}D_p\tilde{g}^{nq}\right]. \tag{A3.34}$$

If we select the value of a_1 according to the correspondence principle,

$$a_1=-\frac{1}{32\pi}, \tag{A3.35}$$

and carry out certain transformations in (A3.34), L_g can be written in the form (8.26).

In conclusion let us find the explicit form of $\delta_\varepsilon L_g$. Substituting into (A3.18) the values of the coefficients given by (A3.26), (A3.33), and (A3.35), we obtain

$$\delta_\varepsilon L_g=D_kQ^k(x),$$

where

$$Q^k(x)=-\frac{1}{32\pi}\left[Q^k_{(1)}+\frac{1}{2}Q^k_{(2)}+\frac{1}{4}Q^k_{(3)}\right].$$

If we now allow for the formulas for $Q^k_{(1)}$, $Q^k_{(2)}$, and $Q^k_{(3)}$, (A3.6), we get

$$Q^k(x)=-\varepsilon^kL_g-\frac{1}{16\pi}\left[D_m\varepsilon^l(x)D_l\tilde{g}^{mk}+\varepsilon^k(x)D_mD_n\tilde{g}^{mn}-D_l\left(\varepsilon^l(x)D_m\tilde{g}^{mk}\right)\right], \tag{A3.36}$$

with L_g given by (A3.34).

Appendix 4

In this Appendix we present the calculations omitted in Chapter 18 when we proceeded from the general formulas (18.63), (18.75), (18.93), and (18.111) to the respective approximate expressions.

A4.1 Deflection of Light and Radio Signals in the Gravitational Field of the Sun

In Chapter 18 we arrived at the general formula (18.63) for $\Delta\varphi$. Let us expand the right-hand side of this formula for $\sqrt{W_0} \gg 2GM_\odot$ in powers of $2GM_\odot/\sqrt{W_0}$ up to second order inclusive. If for the value of J_0 in (18.63) we take

$$J_0 = \frac{(\sqrt{W_0})^{3/2}}{(\sqrt{W_0}-2GM_\odot)^{1/2}},$$

then for factor in front of $F(\nu, q)$ we get

$$\frac{4J_0}{\sqrt{W_0}}\left(\frac{\sqrt{W_0}-2GM_\odot}{\sqrt{W_0}+6GM_\odot}\right)^{1/4} \simeq 4\left(1-\frac{GM_\odot}{\sqrt{W_0}}+\frac{11}{2}\left(\frac{GM_\odot}{\sqrt{W_0}}\right)^2\right)+O\left(\left(\frac{GM_\odot}{\sqrt{W_0}}\right)^3\right). \quad \text{(A4.1)}$$

Since for large values of $\sqrt{W_0}$ (see (18.57) and (18.58))

$$W_1 \simeq -\sqrt{W_0}-2GM_\odot+O\left(\left(\frac{GM_\odot}{\sqrt{W_0}}\right)^3\sqrt{W_0}\right) \quad \text{(A4.2)}$$

and

$$W_2 \simeq 2GM_\odot+O\left(\left(\frac{GM_\odot}{\sqrt{W_0}}\right)^3\sqrt{W_0}\right), \quad \text{(A4.3)}$$

(18.61) and (18.62) yield, respectively,

$$\nu = \frac{\pi}{4}+\frac{3}{2}\frac{GM_\odot}{\sqrt{W_0}}-\frac{3}{2}\left(\frac{GM_\odot}{\sqrt{W_0}}\right)^2+O\left(\left(\frac{GM_\odot}{\sqrt{W_0}}\right)^3\right) \quad \text{(A4.4)}$$

and

$$q^2 = \frac{4GM_\odot}{\sqrt{W_0}}\left(1-\frac{3GM_\odot}{\sqrt{W_0}}\right)+O\left(\left(\frac{GM_\odot}{\sqrt{W_0}}\right)^3\right). \quad \text{(A4.5)}$$

Now let us employ the well-known series expansion of elliptic integrals of the first kind (Abramowitz and Stegun, 1964, and Erdélyi, 1955):

$$F(\nu, q) = \sum_{n=0}^{\infty}\frac{\Gamma(n+1/2)}{n!\,\Gamma(1/2)}q^{2n}S_{2n}(\nu), \quad \text{(A4.6)}$$

where

$$S_{2n}(\nu) = \frac{1}{2^{2n}}\left[\binom{2n}{n}\nu+\sum_{m=1}^{n}(-1)^m\binom{2n}{n-m}\frac{\sin 2m\nu}{m}\right]. \quad \text{(A4.7)}$$

This combined with (A4.4) and (A4.5) yields the following expansion of $F(\nu, q)$ in powers of $GM_\odot/\sqrt{W_0}$ up to second order inclusive:

$$F(\nu, q) \simeq \frac{\pi}{4}+\frac{GM_\odot}{\sqrt{W_0}}\left(1+\frac{\pi}{4}\right)-\frac{3\pi}{16}\left(\frac{GM_\odot}{\sqrt{W_0}}\right)^2+O\left(\left(\frac{GM_\odot}{\sqrt{W_0}}\right)^3\right). \quad \text{(A4.8)}$$

Substitution of (A4.1) and (A4.8) into (18.63) yields

$$\Delta\varphi = \pi + \frac{4GM_\odot}{\sqrt{\widetilde{W}_0}} + 4\left(\frac{GM_\odot}{\sqrt{\widetilde{W}_0}}\right)^2\left(\frac{15\pi}{16} - 1\right). \tag{A4.9}$$

A4.2 The Mercury Perihelion Shift

Let us find the series expansion of the right-hand side of (18.75) in powers of $GM_\odot$ up to second order inclusive. Allowing for the fact that $\sqrt{W_\pm} \gg 2GM_\odot$ in the given problem, we find that (18.74) yields

$$\widetilde{W}_0 \simeq 2GM_\odot\left[1 + 2GM_\odot\left(\frac{1}{\sqrt{W_+}} + \frac{1}{\sqrt{W_-}}\right)\right] + O((GM_\odot)^3), \tag{A4.10}$$

and for the expression standing in (18.75) in front of $F(\pi/2, q)$ we obtain

$$\left[\frac{2\widetilde{W}_0\sqrt{W_+}}{GM_\odot(\sqrt{W_+} - \widetilde{W}_0)}\right]^{1/2} \simeq 2\left[1 + \frac{2GM_\odot}{\sqrt{W_+}} + \frac{GM_\odot}{\sqrt{W_-}} + \frac{6(GM_\odot)^2}{\sqrt{W_+W_-}}\right.$$
$$\left. + 6\left(\frac{GM_\odot}{\sqrt{W_+}}\right)^2 + \frac{3}{2}\left(\frac{GM_\odot}{\sqrt{W_-}}\right)^2\right] + O((GM_\odot)^3). \tag{A4.11}$$

If in (18.76) we allow for (A4.10), it is easy to establish that

$$q^2 \simeq \left(\frac{2GM_\odot}{\sqrt{W_-}} - \frac{2GM_\odot}{\sqrt{W_+}} + \frac{4(GM_\odot)^2}{\sqrt{W_+W_-}} + 4\left(\frac{GM_\odot}{\sqrt{W_-}}\right)^2 - 8\left(\frac{GM_\odot}{\sqrt{W_+}}\right)^2\right) + O((GM_\odot)^3). \tag{A4.12}$$

Then, using the well-known series expansion of complete elliptic integrals of the first kind (Abramowitz and Stegun, 1964, and Erdélyi, 1955),

$$F\left(\frac{\pi}{2}, q\right) = \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 q^2 + \left(\frac{1\times 3}{2\times 4}\right)^2 q^4 + \ldots + \left(\frac{(2n-1)!!}{2^n n!}\right)^2 q^{2n} + \ldots\right], \tag{A4.13}$$

we obtain

$$F\left(\frac{\pi}{2}, q\right) \simeq \frac{\pi}{2}\left[1 + \frac{GM_\odot}{2\sqrt{W_-}} - \frac{GM_\odot}{2\sqrt{W_+}} - \frac{1}{8}\frac{(GM_\odot)^2}{\sqrt{W_+W_-}} + \frac{25}{16}\left(\frac{GM_\odot}{\sqrt{W_-}}\right)^2\right.$$
$$\left. - \frac{23}{16}\left(\frac{GM_\odot}{\sqrt{W_+}}\right)^2\right] + O((GM_\odot)^3). \tag{A4.14}$$

Substituting (A4.11) and (A4.14) into (18.75), we find the sought expansion of $\varphi(r_+) - \varphi(r_-)$:

$$\varphi(r_+) - \varphi(r_-) \simeq \pi\left[1 + \frac{3GM_\odot}{2}\left(\frac{1}{\sqrt{W_+}} + \frac{1}{\sqrt{W_-}}\right)\right.$$
$$\left. + \frac{57(GM_\odot)^2}{16}\left(\frac{1}{W_+} + \frac{1}{W_-}\right) + \frac{51}{8}\frac{(GM_\odot)^2}{\sqrt{W_+W_-}}\right] + O((GM_\odot)^3). \tag{A4.15}$$

A4.3 Time Delay of Radio Signals in the Gravitational Field of the Sun

In experiments conducted by I. I. Shapiro and his group (Shapiro, 1964, and Shapiro, 1979) the first measurements of the time delay of radio signals in the gravitational field of the Sun were performed. In these experiments the reflector was the surface of Mercury in the superior conjunction. Hence, the experimental conditions were such that

$$r_e \gg r_0, \quad r_\mu \gg r_0, \tag{A4.16}$$

where r_0 is a distance of the order of the Sun's radius, and r_e and r_μ are distances from the center of the Sun to the Earth and Mercury, respectively. Since because of (A4.16) we must put

$$W_e = W(r_e) \gg W_0, \quad W_\mu = W(r_\mu) \gg W_0, \tag{A4.17}$$

and

$$W_0 = W(r_0) \gg GM_\odot, \tag{A4.18}$$

in formulas (18.88)-(18.91) we can use expansions in powers of $GM_\odot$.

On the basis of (A4.2) and (A4.3) combined with (18.92) we can find the following approximate expressions for $\nu_{e,\mu}$:

$$\nu_{e,\mu} \simeq \frac{\pi}{4} + \frac{3GM_\odot}{2\sqrt{W_0}} - \frac{1}{2}\frac{\sqrt{W_0}}{\sqrt{W_{e,\mu}}} - \frac{1}{2}\frac{GM_\odot}{\sqrt{W_{e,\mu}}} + O\left(\left(\frac{GM_\odot}{\sqrt{W_0}}\right)^2\right), \tag{A4.19}$$

while for q^2, which enters into (18.88)-(18.91), we already have the approximate expression (A4.5). Note that the denominator of (18.88) contains the difference $W_2 - 2GM_\odot$, which vanishes if for W_2 we take expansion (A4.3). Hence, to calculate the asymptotic value of $I_3(r_{e,\mu})$, we must take an expansion of W_2 containing the next higher-order term. Equation (18.58) yields

$$W_2 \simeq 2GM_\odot + 8\frac{(GM_\odot)^3}{W_0} + O((GM_\odot)^4). \tag{A4.20}$$

If for $F(\nu, q)$ we use representation (A4.6), for $\Pi(\nu, \sigma, q)$ with $|\sigma| < 1$ an expansion of the form (Abramowitz and Stegun, 1964, and Erdélyi, 1955)

$$\Pi(\nu, \sigma, q) = \sum_{n=0}^{\infty} (-\sigma)^n \, B_n^{(-1/2)}\left(\frac{q^2}{\sigma}\right) S_{2n}(\nu), \tag{A4.21}$$

with $S_{2n}(\nu)$ specified by (A4.7), and

$$B_n^{(-1/2)}(Z) = \sum_{m=0}^{n} \binom{-1/2}{m} Z^m, \tag{A4.22}$$

and employ (A4.2) and (A4.20), then, after relatively simple but somewhat cumbersome calculations, we arrive at the following asymptotic expression for $I_i(r_{e,\mu})$, $i = 0, 1, 2, 3$:

$$I_0(r_{e,\mu}) \simeq \sqrt{W_{e,\mu} - W_0} + GM_\odot\left(\frac{\sqrt{W_{e,\mu}} - \sqrt{W_0}}{\sqrt{W_{e,\mu}} + \sqrt{W_0}}\right)^{1/2} + O\left(\left(\frac{GM_\odot}{\sqrt{W_0}}\right)^2\right), \tag{A4.23}$$

$$I_1(r_{e,\mu}) \simeq \ln\frac{\sqrt{W_{e,\mu}} + \sqrt{W_{e,\mu} - W_0}}{\sqrt{W_0}} + O\left(\frac{GM_\odot}{\sqrt{W_0}}\right), \tag{A4.24}$$

$$I_2(r_{e,\mu}) \simeq O\left(\frac{1}{\sqrt{W_0}}\right), \tag{A4.25}$$

$$I_3(r_{e,\mu}) \simeq O\left(\frac{1}{W_0}\right). \tag{A4.26}$$

In deriving (A4.23) and (A4.24) we have employed a relation (Abramowitz and Stegun, 1964, and Erdélyi, 1955) expressing the elliptic integral of the third kind, $\Pi(\nu, \sigma, q)$, with $\sigma > 1$ in terms of $\Pi(\nu, k < 1, q)$, $F(\nu, q)$, and other known functions:

$$\Pi(\nu, \sigma, q) = \frac{1}{(\sigma k)^{1/2}}\left\{\left[\frac{(1-k)k}{k-q^2}\right]^{1/2}\Pi(\nu, k, q) + \frac{q^2}{(\sigma k)^{1/2}}F(\nu, q) + \tan^{-1}\left[\frac{1}{2}(\sigma k)^{1/2}\frac{\sin 2q}{(1+\sigma q^2)^{1/2}}\right]\right\}, \tag{A4.27}$$

with $k = (\sigma + q^2)/(1 + \sigma)$. Substituting (A4.23)-(A4.26) into (18.93), we arrive at a series expansion for $t(r_e, r_\mu) = t(r_0, r_e) + t(r_0, r_\mu)$ in the first order in $GM_\odot$ inclusive:

$$t(r_e, r_\mu) = \sqrt{W_e - W_0} + \sqrt{W_\mu - W_0} + GM_\odot\Big[2\ln\frac{\sqrt{W_\mu}+\sqrt{W_\mu - W_0}}{\sqrt{W_e}-\sqrt{W_e - W_0}} + \left(\frac{\sqrt{W_\mu}-\sqrt{W_0}}{\sqrt{W_\mu}+\sqrt{W_0}}\right)^{1/2} + \left(\frac{\sqrt{W_e}-\sqrt{W_0}}{\sqrt{W_e}+\sqrt{W_0}}\right)^{1/2}\Big]. \quad (A4.28)$$

A4.4 Period of Revolution of a Test Body in Orbit

In Chapter 18 we established a general formula, (18.111), for the period of revolution of a test body in orbit. Let us find the expansion of the right-hand side of this formula in powers of $\sqrt{Gm}$ up to first order inclusive for

$$\sqrt{W_\pm} \gg Gm, \quad \sqrt{W_1} \gg GM. \quad (A4.29)$$

Allowing for the expansions (A4.6), (A4.7), (A4.13), (A4.21), (A4.22), and (A4.27) and the representation of complete elliptic integrals of the third kind in the form of the series expansion

$$\Pi\left(\frac{\pi}{2}, \sigma, q\right) = \sum_{n=0}^{\infty} \frac{\Gamma(n+1/2)}{n!\,\Gamma(1/2)} (-\sigma)^n B_n^{(-1/2)}\left(\frac{q}{\sigma}\right)^2 \quad (A4.30)$$

for $|\sigma| < 1$ and $|q| < 1$, and calculating $\Pi(\pi/2, \sigma, q)$ for $\sigma > 1$ according to formula (A4.27) with $\nu = \pi/2$, we arrive at the following expressions for $\tilde{I}_i(r_+, r_-)$ and $\tilde{I}_i(r_1, r_-)$, $i = 0, 1, 2, 3$:

$$\tilde{I}_0(r_+, r_-) \simeq \frac{\pi}{2}\left(\sqrt{W_+} + \sqrt{W_-}\right) + \frac{\pi}{2}Gm, \quad (A4.31)$$

$$\tilde{I}_1(r_+, r_-) \simeq \pi + Gm\frac{\pi}{2\left(\sqrt{W_+ W_-}\right)^{1/2}}, \quad (A4.32)$$

$$\tilde{I}_2(r_+, r_-) \simeq \frac{\pi}{\left(\sqrt{W_+ W_-}\right)^{1/2}} + O(Gm), \quad (A4.33)$$

$$\tilde{I}_3(r_+, r_-) \simeq \frac{\pi\left(\sqrt{W_+} + \sqrt{W_-}\right)}{2\left(\sqrt{W_+ W_-}\right)^{3/2}} + O(Gm), \quad (A4.34)$$

$$\tilde{I}_0(r_1, r_-) \simeq -\left[\left(\sqrt{W_+} - \sqrt{W_1}\right)\left(\sqrt{W_1} - \sqrt{W_-}\right)\right]^{1/2} + \frac{1}{2}\left[\sqrt{W_+} + \sqrt{W_-} + 2Gm\right] \times\left[\frac{\pi}{2} - \sin^{-1}\left(1 - 2\frac{\sqrt{W_1} - \sqrt{W_-}}{\sqrt{W_+} - \sqrt{W_-}}\right)\right], \quad (A4.35)$$

$$\tilde{I}_1(r_1, r_-) \simeq \left[\frac{\pi}{2} - \sin^{-1}\left(1 - 2\frac{\sqrt{W_1} - \sqrt{W_-}}{\sqrt{W_+} - \sqrt{W_-}}\right)\right] + O(Gm), \quad (A4.36)$$

$$\tilde{I}_2(r_1, r_-) \simeq \frac{1}{\left(\sqrt{W_+ W_-}\right)^{1/2}}\left[\frac{\pi}{2} + \sin^{-1}\left(1 - 2\frac{\sqrt{W_-}}{\sqrt{W_1}}\,\frac{\sqrt{W_+} - \sqrt{W_1}}{\sqrt{W_+} - \sqrt{W_-}}\right)\right] + O(Gm), \quad (A4.37)$$

$$\tilde{I}_3(r_1, r_-) \simeq \frac{\left[\left(\sqrt{W_+} - \sqrt{W_1}\right)\left(\sqrt{W_1} - \sqrt{W_-}\right)\right]^{1/2}}{\sqrt{W_+ W_- W_1}} + \frac{\sqrt{W_+} + \sqrt{W_-}}{2\left(\sqrt{W_+ W_-}\right)^{3/2}}\left[\frac{\pi}{2} + \sin^{-1}\left(1 - 2\frac{\sqrt{W_-}}{\sqrt{W_1}}\,\frac{\sqrt{W_+} - \sqrt{W_1}}{\sqrt{W_+} - \sqrt{W_-}}\right)\right] + O(Gm). \quad (A4.38)$$

Substituting these expressions into (18.111), bearing in mind that

$$\left(\frac{(\sqrt{W_+}+\sqrt{W_-})(\sqrt{W_+}-2Gm)(\sqrt{W_-}-2Gm)}{2Gm(\sqrt{W_+W_-}-2Gm\sqrt{W_+}-2Gm\sqrt{W_-})}\right)^{1/2} \simeq \left(\frac{\sqrt{W_+}+\sqrt{W_-}}{2Gm}\right)^{1/2}, \qquad (A4.39)$$

and retaining terms whose order is no higher than the first in $\sqrt{Gm}$, we arrive at the following approximate expression for T:

$$T \simeq \pi\frac{(\sqrt{W_+}+\sqrt{W_-})^{3/2}}{\sqrt{2Gm}}\left\{1+6\frac{Gm}{\sqrt{W_+}+\sqrt{W_-}}+\frac{1}{\pi}\,\frac{[(\sqrt{W_+}-\sqrt{W_1})(\sqrt{W_1}-\sqrt{W_-})]^{1/2}}{\sqrt{W_+}+\sqrt{W_-}}\right.$$
$$\left.-\frac{1}{2\pi}\left[\frac{\pi}{2}-\sin^{-1}\left(1-2\frac{\sqrt{W_1}-\sqrt{W_-}}{\sqrt{W_+}+\sqrt{W_-}}\right)\right]\right\}+O(Gm). \qquad (A4.40)$$

Appendix 5

In RTG the gravitational field is a physical field and acts on test particles and light. Hence, just like any other field, it does not take the world lines of particles and light outside the causality cone in the Minkowski space-time. In other words, only those gravitational fields, that is, solutions of the equations of motion for the gravitational field, are physical that do not accelerate particles to velocities greater than that of light in an inertial reference frame. This means that for causally related events to take place in the effective Riemann space-time ($ds^2 \geqslant 0$), $d\sigma^2$ must be always nonnegative. This requirement leads to a situation in which the metric coefficients $g_{ik}(x)$ in Galilean coordinates in any inertial reference frame must satisfy the following inequality:

$$g_{00}(x) + 2g_{0\alpha}(x)\,e^{\alpha} + g_{\alpha\beta}(x)\,e^{\alpha}e^{\beta} \leqslant 0, \qquad (A5.1)$$

with $e^{\alpha} = v^{\alpha}/|\mathbf{v}|$ ($\alpha = 1, 2, 3$) the unit velocity vector in three-dimensional Euclidean space in Cartesian coordinates. It is easy to notice that in this case the world lines of particles and light lying in the region $ds^2 \geqslant 0$ of the effective Riemann space-time are sure to lie inside the light cone of the Minkowski space-time.

In Chapter 16, when studying a homogeneous and isotropic universe, we ignored the question of whether the obtained solutions are physical, that is, whether or not they obey the general requirement (A5.1). In this Appendix we discuss this and other related questions.

On the basis of the RTG equations (8.37), in Chapter 16 we established that the line element for a homogeneous and isotropic universe (the Friedmann universe), (16.1), can be reduced to (16.53) if represented in terms of spherical coordinates of the Minkowski space-time. In terms of Cartesian coordinates it can be written thus

$$ds^2 = U(t)\,dt^2 - U^{1/3}(t)\,(dx^2 + dy^2 + dz^2). \qquad (A5.2)$$

When deriving formula (16.53) in Chapter 16, we selected the solution to the RTG equations (8.37) in the form

$$V(t) = U^{1/3}(t) \qquad (A5.3)$$

(see (16.33), (16.34), and (16.52)). However, one can easily notice that the function

$$V(t) = \alpha U^{1/3}(t), \qquad (A5.4)$$

with α a positive constant, also satisfies Eqs. (8.37). For solution (A5.4) the line element of the Friedmann universe assumes the form

$$ds^2 = U(t)\,dt^2 - \alpha U^{1/3}(t)\,(dx^2 + dy^2 + dz^2). \tag{A5.5}$$

This readily leads us to condition (A5.1). Combining the standard notation $U(t) = R^6(t)$ with condition (A5.1), we get

$$R^2(t)\,(R^4(t) - \alpha) \leqslant 0. \tag{A5.6}$$

Since in the case of massless graviton the scale factor $R(t)$ changes from 0 to ∞ (see Chapter 16), for every finite positive α there are always values of $R(t)$ for which inequality (A5.6) ceases to be valid. This means that in RTG there can be no homogeneous and isotropic universes in the Minkowski space-time with a massless graviton, since all such universes lead to nonphysical gravitational fields for $R^4(t) > \alpha$.

The situation changes drastically when the graviton has a nonzero rest mass. It was shown* that for solution (A5.4) the range within which the scale factor $R(t)$ may change is also finite:

$$R_{\min} \leqslant R(t) \leqslant R_{\max}, \tag{A5.7}$$

with $R_{\min} \neq 0$ and $R_{\max} < \infty$. Since $R(t) \leqslant R_{\max}$, by selecting an appropriate value for α, say $R^4_{\max} + 1$, we can easily guarantee that inequality (A5.6) is valid. As we see, the essential thing here is that there exists a finite $R_{\max}$, which, in turn, follows from the assumption that the graviton has a nonzero rest mass.

Thus, according to RTG, a homogeneous and isotropic universe in the Minkowski space-time exists only if the graviton has a nonzero rest mass.

It must be emphasized that in RTG the solutions for $g_{pk}(x)$ have physical meaning only if they satisfy condition (A5.1).

Let us check the validity of (A5.1) for several important solutions of RTG equations.

A5.1. In Chapter 12 we found the solution (12.71) to the RTG equations for the metric coefficients of the effective Riemann space-time outside a spherically symmetric and static source. Following Fock, 1959, let us write these coefficients in terms of Galilean coordinates:

$$\begin{gathered} g_{00}(x) = \frac{r - mG}{r + mG}, \qquad g_{0\alpha}(x) = 0, \\ g_{\alpha\beta}(x) = \left(1 + \frac{mG}{r}\right)^2 \gamma_{\alpha\beta} - \frac{r + mG}{r - mG}\,\frac{(mG)^2}{r^4}\,x_\alpha x_\beta, \end{gathered} \tag{A5.8}$$

where $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$, and m is the mass of the source of gravitational field.

Substituting (A5.8) into (A5.1), we obtain

$$\frac{r - mG}{r + mG} - \left(1 + \frac{mG}{r}\right)^2 - \frac{r + mG}{r - mG}\,\frac{(mG)^2}{r^4}\,(x_\alpha e^\alpha)^2 \leqslant 0. \tag{A5.9}$$

Since

$$\frac{r - mG}{r + mG} - \left(1 + \frac{mG}{r}\right)^2 = -\frac{2mG}{r + mG} - \frac{2mG}{r} - \frac{(mG)^2}{r^2} < 0,$$

inequality (A5.9) for solution (A5.8) outside the source is always valid. Hence, the gravitational field found from (A5.8) on the basis of (8.2) is a physical field outside matter if the radius of the source, r, is greater than mG.

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*Mestvirishvili, M. A., and Yu. V. Chugreev, 1988, *The Friedmann Model of the Evolution of the Universe* (Moscow: Moscow Univ. Press) (in Russian).

A5.2. Using the solution (15.42) found in the weak-field approximation, we can write the $g_{pk}(x)$ in Cartesian coordinates as follows:

$$g_{pk} = \gamma_{pk} + h_{pk}, \tag{A5.10}$$

where

$$h_{pk} = -\Phi_{pk} + \frac{1}{2}\gamma_{pk}\Phi^n_n. \tag{A5.11}$$

Since in TT gauge $\Phi^n_n = 0$ and $\Phi^{0\alpha} = 0$, for the h_{pk} we have

$$h_{00} = h_{0\alpha} = 0, \quad \gamma^{\alpha\beta}h_{\alpha\beta} = 0, \tag{A5.12}$$

$$\tilde{e}^{\alpha}h_{\alpha\beta} = 0, \tag{A5.13}$$

$$\partial^{\alpha}h_{\alpha\beta} = 0. \tag{A5.14}$$

Here $\tilde{e}^{\alpha} = x^{\alpha}/|\mathbf{r}|$.

Substituting (A5.10) into (A5.1) yields

$$e^{\alpha}e^{\beta}h_{\alpha\beta} \leqslant 0. \tag{A5.15}$$

Let us show that if we adhere to the degree of accuracy accepted in Chapter 15, the left-hand side of (A5.15) vanishes. Indeed, since

$$\frac{d\mathbf{r}}{dt} = \frac{d}{dt}(\tilde{\mathbf{e}}r) = \frac{d\tilde{\mathbf{e}}}{dt}r + \tilde{\mathbf{e}}\frac{dr}{dt} = \mathbf{e}v,$$

we have

$$e^{\alpha} = \frac{1}{v}\left(\frac{d\tilde{e}^{\alpha}}{dt}r + \tilde{e}^{\alpha}\frac{dr}{dt}\right). \tag{A5.16}$$

Substituting this into the left-hand side of inequality (A5.15) and allowing for (A5.13), we obtain

$$e^{\alpha}e^{\beta}h_{\alpha\beta} = \frac{r^2}{v^2}\frac{d\tilde{e}^{\alpha}}{dt}\frac{d\tilde{e}^{\beta}}{dt}h_{\alpha\beta} = -\frac{r^2}{v^2}\frac{d\tilde{e}^{\beta}}{dt}\tilde{e}^{\alpha}\frac{dh_{\alpha\beta}}{dt}. \tag{A5.17}$$

But to within terms of the order of r^{-2} and higher we have

$$\tilde{e}^{\alpha}\frac{dh_{\alpha\beta}}{dt} = \tilde{e}^{\alpha}\frac{\partial h_{\alpha\beta}}{\partial t}(1 + v^{\nu}\tilde{e}_{\nu}) \simeq (1 + v^{\nu}\tilde{e}_{\nu})\,\partial^{\alpha}h_{\alpha\beta}.$$

If we now allow for (A5.14), we find that $\tilde{e}^{\alpha}(dh_{\alpha\beta}/dt) = 0$. Hence $e^{\alpha}e^{\beta}h_{\alpha\beta} = 0$, that is what we set out to prove.

A5.3. In Chapter 17, in Cartesian coordinates we arrived at formulas (17.72)-(17.75) for the metric coefficients $g_{pk}(x)$ in the post-Newtonian approximation. Substituting these expressions into (A5.1), we find that

$$-6U + 2U^2 - 4\Phi_1 - 4\Phi_2 - 2\Phi_3 - 6\Phi_4 - G\frac{\partial^2}{\partial t^2}\int \rho(\mathbf{x}', t)\,|\mathbf{x} - \mathbf{x}'|\,d^3x' + 8\gamma_{\alpha\beta}V^{\beta}e^{\alpha} \leqslant 0.$$

Since on the left-hand side of this inequality only the first term is of the order of ε^2 while the rest are of the order of ε^3 and higher, the sign of the left-hand side is determined by the sign of potential U. But U is positive, since $\rho(\mathbf{x}, t) > 0$ (see (17.56)). Hence, inequality (A5.1) is satisfied for solution (17.72)-(17.75) and the respective gravitational field is physical.

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